MST204 HB

The Open University

Mathematics/ Science/ Technology

An Inter-faculty Second Level Course

Mathematical Models and Methods

mathematical models and methods

Handbook

SUP 25512 4

Contents

1 Greek alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	$\sigma\,\varsigma$	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ε	E	epsilon	ν	N	nu	ϕ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
$\theta\,\vartheta$	Θ	theta	π	Π	pi	ω	Ω	omega

2 SI units (Système International d'Unités)

The SI system is based on six base-units, of which the following four are used in this course:

length:	the metre	(abbreviated to m);
mass:	the kilogram	(abbreviated to kg);
time:	the second	(abbreviated to s);
temperature:	the kelvin	(abbreviated to K).

Of these, only the kelvin may be unfamiliar. To express a temperature in kelvins, add 273.2 to temperature expressed in degrees Celsius or 'centigrade' (so the temperature of melting ice is 273.2 K and that of boiling water is 373.2 K). The **absolute zero** of temperature is 0 K.

Units for other quantities are given in terms of the base-units, as in the following examples:

velocity:	metres per second	($\mathrm{m\,s^{-1}}$);
acceleration:	metres per second per second	($\mathrm{m\,s^{-2}}$);
angular frequency:	radians per second	($\mathrm{rad\,s^{-1}}$).

Notice that 'per' is represented by an index -1, and that the usual laws of indices apply, so that 'metres per second per second' becomes $\mathrm{m\,s^{-1}\,s^{-1}} = \mathrm{m\,s^{-2}}$.

The following combinations of base-units occur commonly enough to have special names:

force:	one newton (N)	=	one kilogram-metre per second per second	($\mathrm{kg\,m\,s^{-2}}$);
energy:	one joule (J)	=	one newton-metre	($\mathrm{kg\,m^2\,s^{-2}}$);
power:	one watt (W)	=	one joule per second	($\mathrm{kg\,m^2\,s^{-3}}$).

Other units are often expressed in terms of these subsidiary units, e.g.

stiffness of a spring (see *Unit 7*):	newtons per metre	($\mathrm{N\,m^{-1}}$);
dashpot constant (see *Unit 8*):	newtons per (metre per second)	($\mathrm{N\,m^{-1}\,s}$).

To avoid very large or very small numbers we also use multiple and fractional units, for example the kilometre (km) which is 1000 metres. The most important prefixes for forming these are given below.

giga $= 1\,000\,000\,000 = 10^9$	(abbreviated to G)
mega $= 1\,000\,000 = 10^6$	(abbreviated to M)
kilo $= 1000 = 10^3$	(abbreviated to k)
milli $= 1/1000 = 10^{-3}$	(abbreviated to m)
micro $= 1/1\,000\,000 = 10^{-6}$	(abbreviated to μ)
nano $= 1/1\,000\,000\,000 = 10^{-9}$	(abbreviated to n)

So, for example, the pressure of the atmosphere, which is about $10^5\,\mathrm{N\,m^{-2}}$, is more conveniently written $100\,\mathrm{kN\,m^{-2}}$, i.e. 100 kilonewtons per square metre.

3 Algebra

3.1 Polynomials

A **polynomial** of **degree n** is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a positive integer, x is a variable and $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$. A **linear** polynomial has $n = 1$, a **quadratic** polynomial has $n = 2$ and a **cubic** polynomial has $n = 3$.

The **roots** of a polynomial $p(x)$ are the solutions of the equation $p(x) = 0$. The number of distinct roots is (at most) equal to the degree of the polynomial. If a polynomial $p(x)$ has a root a, then it has a factor $x - a$. If it has a pair of complex conjugate roots a and $\overline{a}$, then it has a real quadratic factor $(x - a)(x - \overline{a}) = x^2 + bx + c$, where $b = -a - \overline{a}$ and $c = a\overline{a}$.

To **factorize** a polynomial is to express it as a product of two or more polynomials of lower degree. The factors of the **difference of two squares** $x^2 - a^2$ are $(x - a)(x + a)$; those of the **perfect square** $x^2 + 2ax + a^2$ are $(x + a)(x + a)$, i.e. $(x + a)^2$.

Completing the square means writing a quadratic polynomial as a sum or difference of perfect squares, as follows:

$$ax^2 + bx + c = a(x + b/2a)^2 \pm k^2,$$

where $k = \sqrt{\pm(c - b^2/4a)}$, the sign being chosen to make k real.

3.2 Summation notation

If $a_1, a_2, \ldots, a_n$ is a sequence of numbers, then

$$\sum_{i=1}^{n} a_i \quad \text{means} \quad a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

Further, if m is an integer less than n, then

$$\sum_{i=m}^{n} a_i \quad \text{means} \quad a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

3.3 Factorials and the binomial theorem

For any positive integer n, we define **n factorial**, written $n!$, by

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n.$$

The first few are $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$. We also define $0! = 1$.

The **binomial theorem** states that, for any positive integer n,

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots \\ &\quad + \frac{n!}{r!(n-r)!}a^{n-r}b^r + \cdots + nab^{n-1} + b^n \\ &= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}a^{n-r}b^r.\end{aligned}$$

In particular,

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2, \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\end{aligned}$$

3.4 Powers and logarithms

If $a > 0$, the power function a^x has the properties

$$a^0 = 1,$$
$$a^{-x} = 1/a^x,$$
$$a^x \times a^y = a^{x+y},$$
$$a^x / a^y = a^{x-y},$$
$$(a^x)^y = a^{xy} = (a^y)^x,$$
$$a^{x/y} = (a^{1/y})^x = (a^x)^{1/y}.$$

If $a > 0$, $a \neq 1$ and $x > 0$, the logarithm function $\log_a x$ has the properties

$$\log_a 1 = 0,$$
$$\log_a(x^{-1}) = -\log_a x,$$
$$\log_a(xy) = \log_a x + \log_a y,$$
$$\log_a(x/y) = \log_a x - \log_a y,$$
$$\log_a(x^y) = y \log_a x.$$

The power and logarithm functions are inverse functions. Hence

$$x = \log_a y \quad \text{means} \quad y = a^x$$

and

$$x = a^y \quad \text{means} \quad y = \log_a x.$$

3.5 Partial fractions

This technique is useful for breaking up functions of the form $P(x)/Q(x)$, where P and Q are both polynomials, into a sum of parts which can be integrated easily.

A simple case (see *Unit 2*) is the function

$$f(x) = \frac{\alpha x + \beta}{ax^2 + bx + c},$$

where $b^2 - 4ac > 0$. The polynomial $ax^2 + bx + c$ has two real roots: call them x_1 and x_2. Then we have

$$f(x) = \frac{\alpha x + \beta}{a(x - x_1)(x - x_2)}.$$

We assume that numbers N_1 and N_2 exist such that

$$\frac{\alpha x + \beta}{a(x - x_1)(x - x_2)} = \frac{N_1}{x - x_1} + \frac{N_2}{x - x_2} \quad (x \neq x_1, x_2),$$

so that $\alpha x + \beta = aN_1(x - x_2) + aN_2(x - x_1)$.

Equating coefficients of x and the constant terms on both sides of the equation gives

$$\alpha = aN_1 + aN_2,$$
$$\beta = -ax_2N_1 - ax_1N_2,$$

a pair of simultaneous equations which can be solved for N_1 and N_2. So we have

$$f(x) = \frac{N_1}{x - x_1} + \frac{N_2}{x - x_2},$$

which can be integrated using the table of standard integrals.

The *general procedure* for partial fractions is as follows.

1. Let the function to be simplified be $f(x) = P(x)/Q(x)$, where P and Q are polynomials with real coefficients.
2. If $P(x)$ has degree equal to or greater than that of $Q(x)$, bring it to the form
$$P(x) = P_1(x)Q(x) + R(x),$$
where $P_1(x)$ is a polynomial and $R(x)$ is a polynomial of degree lower than that of $Q(x)$. Then
$$f(x) = P_1(x) + \frac{R(x)}{Q(x)}.$$
3. Factorize $Q(x)$ into real linear or quadratic factors.
4. For each non-repeated linear factor $(x - x_i)$ in $Q(x)$, write a term $N_i/(x - x_i)$ in the partial fraction expansion.
5. For each non-repeated quadratic factor $x^2 + b_i x + c_i$ (corresponding to a pair of complex conjugate roots in $Q(x)$), write a term $\dfrac{M_i + L_i x}{x^2 + b_i x + c_i}$ in the partial fraction expansion.
6. For each repeated linear factor $(x - x_i)^{m_i}$ in $Q(x)$, write terms
$$\frac{N_{i,1}}{(x - x_i)} + \frac{N_{i,2}}{(x - x_i)^2} + \cdots + \frac{N_{i,m_i}}{(x - x_i)^{m_i}}.$$
7. For each repeated quadratic factor $(x^2 + b_i x + c_i)^{m_i}$, write terms
$$\frac{M_{i,1} + L_{i,1}x}{(x^2 + b_i x + c_i)} + \cdots + \frac{M_{i,m_i} + L_{i,m_i}x}{(x^2 + b_i x + c_i)^{m_i}}.$$
8. Equate the sum of partial fractions to $R(x)/Q(x)$ and multiply both sides of the equation by $Q(x)$. Then equate the coefficients of like powers of x on both sides, to obtain a set of simultaneous equations for all the unknown constants $N_1, N_2, \ldots, M_1, M_2, \ldots, L_1, L_2, \ldots$.

3.6 Rules of algebra

In the following summary of the rules or **axioms** of algebra, the symbols a, b, c stand for arbitrary elements of some set S (which can be the set of, for example, real numbers, complex numbers, square matrices, $n \times m$ matrices, or vectors) and the symbol $*$ stands for the binary operation being considered, which may be addition, multiplication, etc.

Definitions

Closure:	If a and b are elements of S, then so is $a * b$.
Associativity:	$(a * b) * c = a * (b * c)$.
Identity:	There is an 'identity' element e such that $a * e = e * a = a$. (For addition it is called 0; for multiplication, 1 or I.)
Inverse:	To each element a there is a unique inverse element $\tilde{a}$ such that $a * \tilde{a} = \tilde{a} * a = e$. (For addition $\tilde{a}$ is called $-a$; for multiplication, a^{-1}.)
Commutativity:	$a * b = b * a$.
Distributivity: (over binary operation $\circ$)	$a * (b \circ c) = (a * b) \circ (a * c)$. (For example, for addition and multiplication of real numbers $a(b + c) = ab + ac$.)

Summary of rules

	Addition (all cases)	Multiplication					
		Real numbers	Complex numbers	Square matrices	Non-square matrices	Vectors: dot product	Vectors: cross product
1. Closure	✓	✓	✓	✓	(iii)	×	✓
2. Associativity	✓	✓	✓	✓	✓	×	×
3. Identity	✓	✓	✓	✓	×	×	×
4. Inverse	✓	(i)	(i)	(ii)	×	×	×
5. Commutativity	✓	✓	✓	×	×	✓	×
6. Distributivity (over addition)	×	✓	✓	✓	✓	✓	✓

Notes

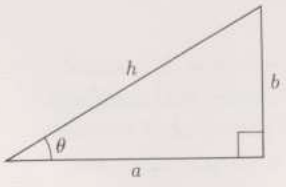
(i) Every element except 0 has an inverse.

(ii) All non-singular square matrices (i.e. those whose rows are linearly independent) have inverses.

(iii) The product of an $n \times m$ and an $r \times s$ matrix (in that order) exists if and only if $m = r$, in which case the product is an $n \times s$ matrix.

4 Trigonometry

4.1 Trigonometric functions and their inverses

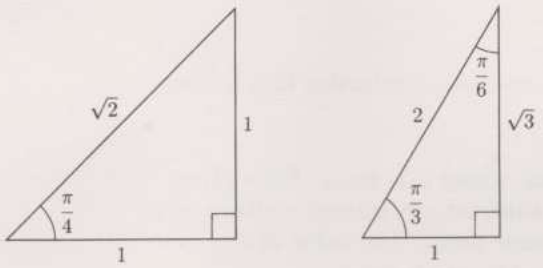


Function	Definition for acute angles in terms of triangle above	Definition in terms of sin and cos	Range of values for function
$\sin\theta$	$\frac{b}{h}$		$-1 \leq \sin\theta \leq 1$
$\cos\theta$	$\frac{a}{h}$		$-1 \leq \cos\theta \leq 1$
$\tan\theta$	$\frac{b}{a}$	$\frac{\sin\theta}{\cos\theta}$	All real values
$\cot\theta$	$\frac{a}{b}$	$\frac{\cos\theta}{\sin\theta}$	All real values
$\sec\theta$	$\frac{h}{a}$	$\frac{1}{\cos\theta}$	$\sec\theta \geq 1$ or ≤ -1
$\operatorname{cosec}\theta$	$\frac{h}{b}$	$\frac{1}{\sin\theta}$	$\operatorname{cosec}\theta \geq 1$ or ≤ -1

Inverse function	Range of values for x	Definition	Range of values* for inverse function
$\arcsin x$	$-1 \leq x \leq 1$	$= \theta$ with $\sin\theta = x$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\arccos x$	$-1 \leq x \leq 1$	$= \theta$ with $\cos\theta = x$	$0 \leq \theta \leq \pi$
$\arctan x$	All real x	$= \theta$ with $\tan\theta = x$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\operatorname{arccot} x$	All real x	$= \theta$ with $\cot\theta = x$	$0 < \theta < \pi$
$\operatorname{arcsec} x$	$\lvert x\rvert \geq 1$	$= \theta$ with $\sec\theta = x$	$0 \leq \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta \leq \pi$
$\operatorname{arccosec} x$	$\lvert x\rvert \geq 1$	$= \theta$ with $\operatorname{cosec}\theta = x$	$-\frac{\pi}{2} \leq \theta < 0$ and $0 < \theta \leq \frac{\pi}{2}$

*Sometimes called the **principal value range** of that inverse function.

4.2 Two useful triangles



From these it can be seen that, for example, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

4.3 Trigonometric identities

Pythagorean relations

$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Addition formulas

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha + \beta) + \frac{1}{2}\sin(\alpha - \beta)$
$\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha + \beta) - \frac{1}{2}\sin(\alpha - \beta)$
$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta)$
$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$

Double angle formulas

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

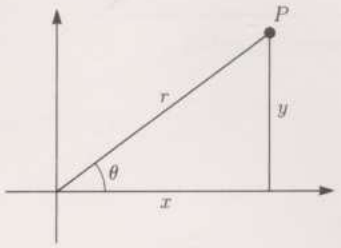
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

4.4 Polar coordinates



The point P whose polar coordinates are $[r,\theta]$ has Cartesian coordinates (x,y), where

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The value of r is always positive (except at the origin, where it is zero). For a given point P the value of θ is not unique: we can add or subtract any integer multiple of 2π and obtain another value for θ which describes the same point. The value of θ satisfying $-\pi < \theta \leq \pi$ is called the **principal value** of θ, and is given by

$$\theta = \begin{cases} \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y \geq 0, \\ -\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0. \end{cases}$$

5 Functions and graphs

5.1 Functions

Any expression or formula which involves a variable (say x), and whose value is uniquely determined by the value of x, is called a **function** of x.

If a variable y is a function of x (i.e. is equal to a function of x) then we call x the **independent variable** and y the **dependent variable**, and we may write $y = y(x)$. Here $y(x)$ stands for the function of x (i.e. for the formula involving x).

A **constant** is a variable whose value remains the same whatever value we give to the independent variable.

It is also possible to think of a function in a more abstract way, as a mathematical object consisting of three parts:

(i) the **domain**, which is the set from which the value of x is taken;

(ii) the **codomain**, which is a set in which the value of y will lie;

(iii) the **rule** which associates to each x in the domain a unique element in the codomain, called the **image** of x.

If no domain and codomain are specified, then the domain is taken to be the largest possible set of real numbers for which the rule makes sense, and the codomain to be the set of all real numbers.

If f and g are two functions, then their **sum** is a function $f+g$ defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{(for all } x\text{)}.$$

Moreover, if A and B are any two numbers, then the function $Af + Bg$ is defined by

$$(Af + Bg)(x) = Af(x) + Bg(x) \quad \text{(for all } x\text{)}.$$

The function $f(g(x)) = f \circ g(x)$ is called the **composite function** or **composition** for the functions f and g.

The **graph** of a function f is the curve in the (x, y)-plane whose equation is $y = f(x)$.

A **constant function** is one that assigns the same image to all the elements in its domain. Its graph is a straight line parallel to the x-axis. A special case is the **zero function**, which assigns the image zero to all the elements in its domain.

A **linear function** is one having the form $ax + b$, where a and b are constants. The constant a is the slope and the constant b is the y-intercept of the graph.

A **continuous** function is one such that a small change in x cannot produce a large change in $f(x)$ (i.e. the graph of f has no breaks or jumps).

The **limit** at a of a function f, written $\lim_{x \to a} f(x)$, is a number L such that we can make $|f(x) - L|$ as small as we please by choosing any x sufficiently close to a. For a continuous function, $\lim_{x \to a} f(x) = f(a)$.

The **limit** for large x of a function f, written $\lim_{x \to \infty} f(x)$, is a number L such that we can make $|f(x) - L|$ as small as we please by choosing any x sufficiently large.

5.2 Graphs of some common functions

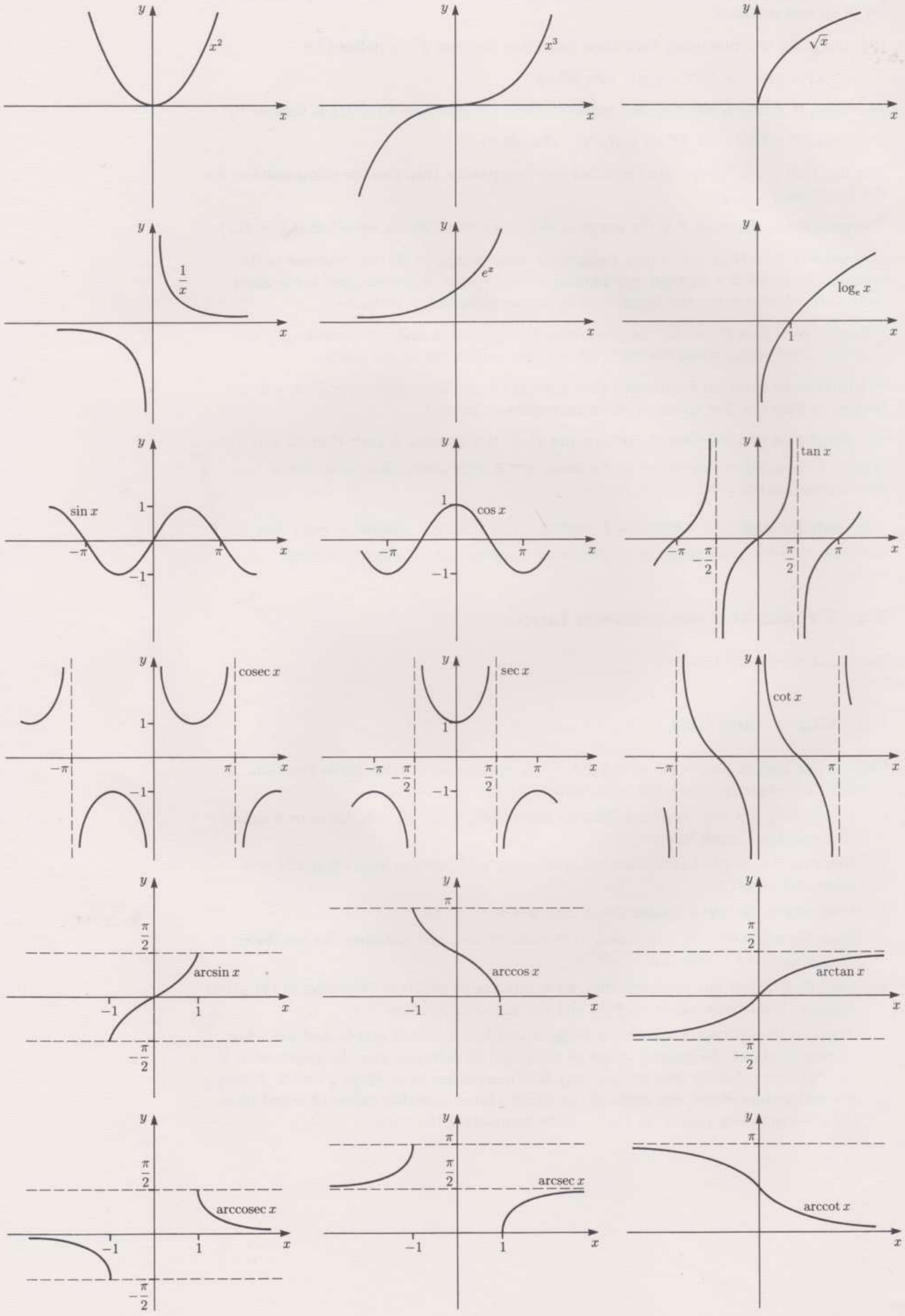
See sketches on page 12.

5.3 Curve sketching

Suppose we wish to sketch the graph $y = f(x)$, where $f(x)$ is some given function. A possible procedure for doing this is given below.

1. Check if $f(x)$ is any standard function whose graph you already know or a simple combination of such functions.
2. Determine how y behaves when x is very large and positive and when x is very large and negative.
3. Find where the curve crosses the x- and y-axes, if at all.
4. Look for any values of x at which $f(x)$ is undefined, and examine the behaviour of $f(x)$ near these values of x.
5. Find if there are any local maxima, local minima or points of inflection in the given domain. Find these values of $f(x)$ and the associated values of x.
6. Transfer the information found in Steps 3 and 5 to a sketch graph, and use what you know about the general shape of the graph to help you join the points with a smooth curve, taking into account any information found in Steps 2 and 4. If you are still unsure about any parts of the curve, choose suitable values of x and plot the corresponding points $(x, f(x))$ before completing the curve.

Sketch graphs of some common functions



6 Differentiation

(Only ordinary differentiation is considered here. For *partial* differentiation, see the summary of *Unit 25.*)

6.1 Notation and terminology

If f is a function, its **derived function** or **derivative** f' is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The process of calculating $f'(x)$ from $f(x)$ is called **differentiation** of $f(x)$ with respect to x. Differentiation with respect to x is denoted by the symbol $\dfrac{d}{dx}$ written to the left of the expression or variable which is differentiated, so that, for example,

$$\frac{df(x)}{dx} \quad \text{means} \quad f'(x),$$

and if y is a variable that stands for $f(x)$, then $\dfrac{dy}{dx}$ is a variable that stands for $f'(x)$.

To save space, we often print dy/dx in place of $\dfrac{dy}{dx}$.

When the independent variable is the letter x, we often use a prime in place of $\dfrac{d}{dx}$, so that y' means $\dfrac{dy}{dx}$. When the independent variable is t (time), we often use a dot in place of $\dfrac{d}{dt}$, so that $\dot{u}$ means $\dfrac{du}{dt}$.

The derivative of a derivative is called a **second derivative**. For example, the second derivative of the function f, denoted by f'', is the (first) derivative of f', defined by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}.$$

The symbol y'' or $\dfrac{d^2y}{dx^2}$ is used for $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$,

and $\ddot{u}$ or $\dfrac{d^2u}{dt^2}$ is used for $\dfrac{d}{dt}\left(\dfrac{du}{dt}\right)$.

Third and **higher derivatives** are defined and written analogously. The nth derivative of f is denoted by $f^{(n)}$.

6.2 Rules of differentiation

Sum rule: if u and v are functions of x, then

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Constant multiplier rule: if k is a constant and u is a function of x, then

$$\frac{d}{dx}(ku) = k\frac{du}{dx}.$$

Product and quotient rules: if u and v are functions of x, then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx},$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{du}{dx}v - u\frac{dv}{dx}\right) \Big/ v^2.$$

Chain rule or **'function of a function' rule:** if f and g are two functions, then the derivative of their composition is given by the **function of a function rule**:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

A more common way to write this is the **chain rule**:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where (in this case) u stands for $g(x)$ and y for $f(u)$.

Implicit differentiation: given an equation connecting two variables, say x and y, we calculate dy/dx by differentiating both sides with respect to x and then solving the resulting equation algebraically for dy/dx (instead of solving for y before differentiating).

6.3 Standard derivatives

In each of the following cases the domain is the largest set of real numbers x for which the function is defined.

Function	Derivative
x^α	$\alpha x^{\alpha-1}$ (α any number)
$\log_e x$	$1/x$
$\log_e(-x)$	$1/x$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

The following derivatives are given in a form that is useful for integration. The constant a is assumed positive, and in the derivatives of arcsec and arccosec it is assumed that $x > a$ (rather than $x < -a$).

Function	Derivative
$\arcsin\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\arccos\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}$
$\arctan\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$
$\operatorname{arccot}\left(\frac{x}{a}\right)$	$\frac{-a}{a^2+x^2}$
$\operatorname{arcsec}\left(\frac{x}{a}\right)$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\operatorname{arccosec}\left(\frac{x}{a}\right)$	$\frac{-a}{x\sqrt{x^2-a^2}}$

6.4 Taylor series

About $x = 0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad (-1 < x < 1),$$

$$(a+x)^r = a^r + ra^{r-1}x + \frac{r(r-1)}{2!}a^{r-2}x^2 + \cdots \quad (-a < x < a)$$

(where r is any real number and a is any positive number).

In general, about $x = 0$

$$f(x) = f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots,$$

and about $x = a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots.$$

The ***n*th-order Taylor approximation** to $f(x)$ about $x = a$ is

$$f(x) \simeq f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a).$$

In particular, $n = 1$ gives the **tangent approximation**

$$f(x) \simeq f(a) + (x-a)f'(a)$$

and $n = 2$ gives the **quadratic Taylor approximation**

$$f(x) \simeq f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a).$$

These approximations are good when x is close to a.

7 Integration

7.1 Standard integrals

The following integrals are obtained by reversing the entries in the tables of standard derivatives. In each case n stands for a non-negative integer, a for a non-zero real constant and α for a real constant.

Function	Integral	Domain
x^n	$\frac{x^{n+1}}{n+1} + C$	All x
$x^\alpha \quad (\alpha \neq -1)$	$\frac{x^{\alpha+1}}{\alpha+1} + C$	$x > 0$ (and $x < 0$ as well if α is an integer)
$\frac{1}{x}$	$\log_e x + C$	$x > 0$
$\frac{1}{x}$	$\log_e(-x) + C$	$x < 0$
e^{ax}	$\frac{1}{a}e^{ax} + C$	All x
$\sin ax$	$-\frac{1}{a}\cos ax + C$	All x
$\cos ax$	$\frac{1}{a}\sin ax + C$	All x
$\sec^2 ax$	$\frac{1}{a}\tan ax + C$	$-\frac{\pi}{2} < ax < \frac{\pi}{2}$
$\operatorname{cosec}^2 ax$	$-\frac{1}{a}\cot ax + C$	$0 < ax < \pi$
$\sec ax \tan ax$	$\frac{1}{a}\sec ax + C$	$-\frac{\pi}{2} < ax < \frac{\pi}{2}$
$\operatorname{cosec} ax \cot ax$	$-\frac{1}{a}\operatorname{cosec} ax + C$	$0 < ax < \pi$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin \frac{x}{a} + C$	$-a < x < a$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan \frac{x}{a} + C$	All x
$\frac{1}{x\sqrt{x^2 - a^2}}$	$\frac{1}{a}\operatorname{arcsec} \frac{x}{a} + C$	$x > a$

The following integrals may also come in useful. As before, the constant a must be non-zero, but b, α, β, γ can take any real values (except where stated otherwise).

Function	Integral	Domain
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e(ax+b)+C$	$ax+b>0$
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e(-ax-b)+C$	$ax+b<0$
$\log_e ax$	$x(\log_e(ax)-1)+C$	$ax>0$
$\tan ax$	$-\frac{1}{a}\log_e(\cos ax)+C$	$-\frac{\pi}{2}<ax<\frac{\pi}{2}$
$\cot ax$	$\frac{1}{a}\log_e(\sin ax)+C$	$0<ax<\pi$
$\sec ax$	$\frac{1}{a}\log_e(\sec ax+\tan ax)+C$	$-\frac{\pi}{2}<ax<\frac{\pi}{2}$
$\operatorname{cosec} ax$	$\frac{1}{a}\log_e(\operatorname{cosec} ax-\cot ax)+C$	$0<ax<\pi$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log_e\left(\frac{a+x}{a-x}\right)+C$	$\lvert x\rvert<\lvert a\rvert$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)+C$	$x>\lvert a\rvert$
	$\frac{1}{2a}\log_e\left(\frac{-x+a}{-x-a}\right)+C$	$x<-\lvert a\rvert$
$\frac{1}{\sqrt{x^2+a^2}}$	$\log_e(x+\sqrt{x^2+a^2})+C$	All x
$\frac{1}{\sqrt{x^2-a^2}}$	$\log_e(x+\sqrt{x^2-a^2})+C$	$x>\lvert a\rvert$
	$\log_e(-x-\sqrt{x^2-a^2})+C$	$x<-\lvert a\rvert$
$\frac{1}{\sqrt{\alpha x^2+2\beta x+\gamma}}$ $(\alpha>0)$	$\frac{1}{\sqrt{\alpha}}\log_e\left(\frac{\alpha x+\beta}{\sqrt{\alpha}}+\sqrt{\alpha x^2+2\beta x+\gamma}\right)+C$	If $\alpha x^2+2\beta x+\gamma$ has real roots, $x>$ the larger root; if not, x is unrestricted.
$\frac{1}{\sqrt{-\alpha x^2+2\beta x+\gamma}}$ $(\alpha>0, \beta^2+\alpha\gamma>0)$	$-\frac{1}{\sqrt{\alpha}}\arccos\frac{\alpha x-\beta}{\sqrt{\beta^2+\alpha\gamma}}+C$	x between the roots of $-\alpha x^2+2\beta x+\gamma$
$\sqrt{x^2+a^2}$	$\frac{a^2}{2}\log_e(x+\sqrt{x^2+a^2})+\frac{x}{2}\sqrt{x^2+a^2}+C$	All x
$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\log_e(x-\sqrt{x^2-a^2})+\frac{x}{2}\sqrt{x^2-a^2}+C$	$x>\lvert a\rvert$
	$\frac{a^2}{2}\log_e(-x+\sqrt{x^2-a^2})+\frac{x}{2}\sqrt{x^2-a^2}+C$	$x<-\lvert a\rvert$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$	$\lvert x\rvert<\lvert a\rvert$

For the following integrals, the domain comprises all real values of x.

Function	Integral
$e^{\alpha x}\sin\beta x$	$\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x-\beta\cos\beta x)+C$
$e^{\alpha x}\cos\beta x$	$\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}(\alpha\cos\beta x+\beta\sin\beta x)+C$
$x\sin ax$	$-\dfrac{x}{a}\cos ax+\dfrac{1}{a^2}\sin ax+C$
$x\cos ax$	$\dfrac{x}{a}\sin ax+\dfrac{1}{a^2}\cos ax+C$
$x^2\sin ax$	$\left(-\dfrac{x^2}{a}+\dfrac{2}{a^3}\right)\cos ax+\dfrac{2x}{a^2}\sin ax+C$
$x^2\cos ax$	$\left(\dfrac{x^2}{a}-\dfrac{2}{a^3}\right)\sin ax+\dfrac{2x}{a^2}\cos ax+C$
$x^n\sin ax$	$-\dfrac{x^n}{a}\cos ax+\dfrac{n}{a}\displaystyle\int x^{n-1}\cos ax\,dx+C$
$x^n\cos ax$	$\dfrac{x^n}{a}\sin ax-\dfrac{n}{a}\displaystyle\int x^{n-1}\sin ax\,dx+C$
$\sin^2 ax$	$\dfrac{1}{2}x-\dfrac{1}{4a}\sin 2ax+C$
$\cos^2 ax$	$\dfrac{1}{2}x+\dfrac{1}{4a}\sin 2ax+C$
$\sin ax\cos ax$	$-\dfrac{1}{4a}\cos 2ax+C$

In the following three cases we require $\alpha\neq\beta$ and $\alpha\neq-\beta$.

Function	Integral
$\sin\alpha x\sin\beta x$	$\dfrac{\sin(\alpha-\beta)x}{2(\alpha-\beta)}-\dfrac{\sin(\alpha+\beta)x}{2(\alpha+\beta)}+C$
$\cos\alpha x\cos\beta x$	$\dfrac{\sin(\alpha-\beta)x}{2(\alpha-\beta)}+\dfrac{\sin(\alpha+\beta)x}{2(\alpha+\beta)}+C$
$\sin\alpha x\cos\beta x$	$-\dfrac{\cos(\alpha-\beta)x}{2(\alpha-\beta)}-\dfrac{\cos(\alpha+\beta)x}{2(\alpha+\beta)}+C$

7.2 Rules of integration

Examples

1. The fundamental theorem of calculus

$$\int f(x)\,dx=F(x)+C$$

if and only if $f(x)=\dfrac{dF(x)}{dx}$.

$$\int x\,dx=\tfrac{1}{2}x^2+C,$$

$$x=\frac{d}{dx}(\tfrac{1}{2}x^2).$$

2. Linearity rule

$$\int[\alpha f(x)+\beta g(x)]\,dx=\alpha\int f(x)\,dx+\beta\int g(x)\,dx.$$

$$\int(\tfrac{1}{2}e^x-\tfrac{1}{2}e^{-x})\,dx=\tfrac{1}{2}\int e^x\,dx-\tfrac{1}{2}\int e^{-x}\,dx=\tfrac{1}{2}e^x+\tfrac{1}{2}e^{-x}+C.$$

3. Substitution rules

For integrals of the form

$$I = \int g(f(x))\frac{df(x)}{dx}\,dx,$$

substitute $u = f(x)$ and use $\dfrac{du}{dx}\,dx = du$ to obtain

$$I = \int g(u)\,du.$$

Otherwise, find a suitable relation between u and x and use

$$dx = \frac{dx}{du}\,du.$$

To find $\displaystyle\int \sin(ax+b)\,dx = I_1$, let $u = ax + b$; then $a\,dx = du$. So

$$\begin{aligned} I_1 &= \frac{1}{a}\int \sin(ax+b)\,a\,dx \\ &= \frac{1}{a}\int \sin u\,du = -\frac{1}{a}\cos u + C \\ &= -\frac{1}{a}\cos(ax+b) + C. \end{aligned}$$

To find $\displaystyle\int \frac{dx}{(1+x^2)^{3/2}} = I_2$, let $x = \tan u$; then $dx = \sec^2 u\,du$. So

$$\begin{aligned} I_2 &= \int \frac{\sec^2 u\,du}{(1+\tan^2 u)^{3/2}} \\ &= \int \frac{\sec^2 u\,du}{\sec^3 u} \\ &= \int \cos u\,du = \sin u + C \\ &= \sin(\arctan x) + C. \end{aligned}$$

4. Integration by parts

Let $u = u(x)$ and $v = v(x)$; then

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

To find $\displaystyle\int x\cos x\,dx = I_3$, let $u = x$ and $v = \sin x$; then

$$\begin{aligned} I_3 &= \int x\frac{d}{dx}(\sin x)\,dx \\ &= x\sin x - \int \sin x \times 1\,dx \\ &= x\sin x + \cos x + C. \end{aligned}$$

7.3 Integration methods

Rational functions

To integrate a rational function, that is, one of the form $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, we first express the function as a sum of partial fractions (see Subsection 3.5). Then integrate each partial fraction as follows:

$\displaystyle\int \frac{dx}{ax+b}$ — a standard form;

$\displaystyle\int \frac{dx}{(ax+b)^n}$ — substitute $u = ax + b$;

$\displaystyle\int \frac{\alpha x+\beta}{(ax^2+bx+c)^n}\,dx$ — with $b^2 < 4ac$, substitute $2ax + b = \sqrt{4ac - b^2}\,\tan u$.

Irrational functions

Integrals containing the expressions $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{x^2 + a^2}$ can often be simplified by the following substitutions:

for $\sqrt{a^2 - x^2}$ use $x = a\sin u$;

for $\sqrt{x^2 - a^2}$ use $x = a\sec u$;

for $\sqrt{x^2 + a^2}$ use $x = a\tan u$.

More generally, for $\sqrt{\alpha x^2 + \beta x + \gamma}$:

if $\alpha > 0$ and $\beta^2 < 4\alpha\gamma$ use $2\alpha x + \beta = \sqrt{4\alpha\gamma - \beta^2}\,\tan u$;

if $\alpha > 0$ and $\beta^2 > 4\alpha\gamma$ use $2\alpha x + \beta = \sqrt{\beta^2 - 4\alpha\gamma}\,\sec u$;

if $\alpha < 0$ and $\beta^2 > 4\alpha\gamma$ use $2\alpha x + \beta = \sqrt{\beta^2 - 4\alpha\gamma}\,\sin u$.

(In the remaining case the square root is always imaginary.)

Exponential and trigonometric functions

For $\int P(x) \begin{Bmatrix} e^{ax} \\ \sin ax \\ \cos ax \end{Bmatrix} dx$, where $P(x)$ is a polynomial, integrate by parts (several times if necessary) with $u = P(x)$, $\dfrac{dv}{dx} = e^{ax}$ or $\sin ax$ or $\cos ax$. See also the 'standard integrals' table for $\int x^n \sin ax\, dx$ and $\int x^n \cos ax\, dx$.

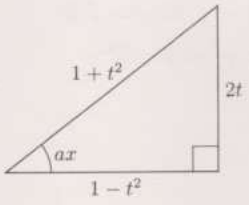
For $\int e^{\alpha x} \begin{Bmatrix} \sin \beta x \\ \cos \beta x \end{Bmatrix} dx$, see the 'standard integrals' table.

For $\int \sin^m ax \cos^n ax\, dx$,

if n is odd use $u = \sin ax$;
if m is odd use $u = \cos ax$;
if m, n are both even, then
if $n < 0$ use $u = \tan ax$,
if $m < 0$ use $u = \cot ax$,
if m, n both ≥ 0 use $\sin^2 ax = \frac{1}{2}(1 - \cos 2ax)$, $\cos^2 ax = \frac{1}{2}(1 + \cos 2ax)$.

For other integrals involving trigonometric functions only, it may be helpful to express them in terms of $\sin ax$ and $\cos ax$ and then use the substitution $t = \tan \frac{1}{2}ax$, according to which

$$\sin ax = \frac{2t}{1+t^2}, \quad \cos ax = \frac{1-t^2}{1+t^2}, \quad a\,dx = \frac{2\,dt}{1+t^2}.$$



For integrals involving e^{ax} only, the substitution $u = e^{ax}$ may be useful.

8 Unit outlines

Unit 1 Recurrence relations

Section 1: Introduction to recurrence relations

1. An equation relating each term in a sequence to one or more previous terms, which can be used to generate the whole sequence once the first few terms are known, is called a **recurrence relation**.

2. A recurrence relation is **of *k*th order** if the difference between the highest and lowest subscripts in it is k. To generate a sequence using such a recurrence relation we need in addition the values of the first k members of the sequence; any set of conditions from which they can be deduced are called the **initial conditions**.

3. A recurrence relation is **linear** if it can be written as

$$u_{r+1} = a_r u_r + b_r u_{r-1} + c_r u_{r-2} + \cdots + p_r,$$

where p_r and the **coefficients** a_r, b_r, c_r, ... may depend on r, but do not depend on any of the us. Otherwise it is said to be **non-linear**.

4. A linear recurrence relation has **constant coefficients** if none of the coefficients a_r, b_r, c_r, ... depend on r. (But p_r may depend on r.)

5. A linear recurrence relation is **homogeneous** if $p_r = 0$ for all values of r. Otherwise it is said to be **non-homogeneous** (or **inhomogeneous**).

6. The **general solution** of a recurrence relation is a formula containing one or more arbitrary constants, from which all the **particular solutions** can be obtained by giving the arbitrary constants particular values which may be determined from the initial conditions.

7. The general solution of the **linear constant-coefficient first-order recurrence relation**

$$u_{r+1} = au_r + p$$

is

$$u_n = \begin{cases} Ba^n - \dfrac{p}{a-1} & \text{if } a \neq 1, \\ A + np & \text{if } a = 1, \end{cases}$$

where A and B are arbitrary constants.

Section 2: Linear second-order recurrence relations

To obtain the general solution of the linear homogeneous constant-coefficient second-order recurrence relation

$$u_{r+1} = au_r + bu_{r-1},$$

we first consider the **auxiliary equation**

$$x^2 = ax + b.$$

There are three cases to consider.

(i) If $a^2 + 4b > 0$, the auxiliary equation has two distinct real solutions, λ and μ, and the general solution for the recurrence relation is

$$u_n = A\lambda^n + B\mu^n.$$

(ii) If $a^2 + 4b = 0$, the auxiliary equation has a repeated real solution λ, and the general solution is

$$u_n = (A + Bn)\lambda^n.$$

(iii) If $a^2 + 4b < 0$, the auxiliary equation has no real solutions. (See *Unit 5*, Section 5.)

Section 3: Numerical difficulties

1. If $\overline{x}$ is an approximation to some quantity whose true value is x, we call $\overline{x} - x$ the **absolute error** in x, and we call $(\overline{x} - x)/x$ the **relative error** in x.

2. Consider any problem in which we are given numerical data and we want to calculate some result from them. The problem is said to be **absolutely ill-conditioned** (with respect to small changes in the data) if a small absolute error in the data gives rise to a significantly larger absolute error in the result. It is said to be **relatively ill-conditioned** if a small relative error in the data gives rise to a significantly larger relative error in the result. It is said to be **absolutely well-conditioned** if the absolute error in the result is smaller than the absolute error in the data, and **relatively well-conditioned** if the relative error in the result is smaller than the relative error in the data.

3. The **scale factor** of a calculation is the factor by which a small error in the data is multiplied to give the corresponding error in the result. If the magnitude of the scale factor is significantly greater than 1, then the problem is absolutely ill-conditioned with respect to small changes in the data; if the magnitude of the scale factor is less than 1, then the problem is absolutely well-conditioned.

4. Consider the problem of calculating u_n from u_0, using the recurrence relation

$$u_{r+1} = au_r + p.$$

The scale factor for this problem is a^n. If the magnitude of a^n is less than 1, the problem is both absolutely and relatively well-conditioned with respect to small changes in u_0. If the magnitude of a^n is significantly larger than 1, then the problem is absolutely ill-conditioned. For any a such that $|a| > 1$, the problem is therefore absolutely ill-conditioned for sufficiently large n. It is relatively ill-conditioned if, in addition, the condition

$$u_0 \simeq -\frac{p}{a-1}$$

is satisfied.

5. If $|a_r| > 1$ for all r, then the problem of calculating u_n from u_0 using the more general recurrence relation

$$u_{r+1} = a_r u_r + p_r$$

is absolutely ill-conditioned (with respect to small changes in u_0) for sufficiently large n. It is relatively ill-conditioned if, in addition, the initial value u_0 is close to a value for which u_n, the true solution of the recurrence relation, remains small for large values of n.

Section 4: Living with ill-conditioning

1. Mortgage problems, relating to a situation where £X are borrowed at $I\%$ per annum interest and repaid by monthly payments of £M, can be modelled using the recurrence relation

$$u_{r+1} = (1 + I/100)u_r - 12M,$$

where u_r is the amount owed at the end of the rth year. The initial condition is $u_0 = X$.

Mortgage problems are often highly sensitive to small changes in the data, particularly the values of I and M.

2. Sometimes ill-conditioning can be cured by a reformulation of the problem. For example, in the recurrence relation

$$u_{r+1} = a_r u_r + p_r,$$

with $|a_r| > 1$ for all values of r, the solution which remains small cannot be reliably calculated by **forward recurrence**, i.e. starting with u_0 and calculating u_1, u_2, ... in succession, because this problem is relatively ill-conditioned. The reformulation which cures the ill-conditioning is to use **backward recurrence** instead. This means rewriting the recurrence relation with u_r as the subject of the formula,

$$u_r = \frac{u_{r+1} - p_r}{a_r},$$

and calculating values of u_r from it working down from a large value of r, say R. Even though we know very little about the value of u_R, it is easy to obtain accurate results because the new problem is very well-conditioned.

Unit 2 Differential equations I

Section 1: Direction fields

1. A **first-order differential equation** is an equation connecting the derivative dy/dx of some unknown function to the variables x and y which are related by that function. By solving the equation algebraically for dy/dx we can bring it to the standard form

$$\frac{dy}{dx} = m(x, y), \qquad (1)$$

where $m(x, y)$ stands for some formula which may contain either or both of the variables x and y.

2. By a **solution** of a differential equation we mean a function—that is, a rule or formula giving y in terms of x—which satisfies the differential equation.

3. A **direction field** is a rule or formula which specifies a slope at each point in the (x, y)-plane (or some part of it). To each direction field there corresponds a differential equation, constructed by taking $m(x, y)$ in (1) to be the formula giving the slope of the direction field at the point (x, y). If $y = f(x)$ is the equation of a **trajectory** of the direction field—that is, a smooth curve at every point of which the slope of the curve matches that of the direction field—then $y = f(x)$ is also a solution of the corresponding differential equation. The direction field has an infinite family of trajectories, each one corresponding to a different solution of the differential equation.

4. For some differential equations, an exact formula can be found for the solutions. This formula, called the **general solution**, contains an arbitrary constant of integration, often denoted by C. Different numerical values for C give different solutions to the differential equation (usually called, in this context, **particular solutions** to distinguish them from the general solution) and therefore label different trajectories of the direction field.

5. Whether or not it is possible to find a formula for the solution of a differential equation, the direction field is a useful source of information about its solution; in particular it can be used to sketch solution curves (trajectories) and so obtain an idea of their qualitative behaviour.

Section 2: A numerical method

Given a differential equation

$$\frac{dy}{dx} = m(x, y)$$

and the condition that $y(x_0) = y_0$, where x_0 and y_0 are specified, then **Euler's method** can be used to find $y(b)$, where b is a given value of x, as follows.

(i) Choose $h = (b - x_0)/n$, where n is the number of steps to be used.

(ii) For $r = 0, 1, \ldots, n - 1$, use the recurrence relation

$$Y_{r+1} = Y_r + h\, m(x_r, Y_r),$$

where $x_r = x_0 + rh$, so that $x_n = b$, and $Y_0 = y_0$.

(iii) Then Y_n is an approximation to $y(b)$.

The error in the approximation to $y(b)$ is roughly proportional to h, and therefore inversely proportional to the cost (i.e. number of steps) of the calculation.

Sections 3 and 4: Exact methods

1. To look for an exact solution to a given differential equation, first bring it to the standard form

$$\frac{dy}{dx} = m(x, y).$$

Then

(i) if $m(x, y)$ has the form

$$m(x, y) = f(x)g(y)$$

(where either or both of $f(x)$ and $g(y)$ may be constants) use separation of variables (see below);

(ii) if $m(x, y)$ has the linear form

$$m(x, y) = k(x) + l(x)y$$

(again either or both of $k(x)$ and $l(x)$ may be constants) use the integrating factor method (see below).

2. Separation of variables (Procedure 3.3)

To solve the differential equation

$$\frac{dy}{dx} = f(x)g(y) \quad (g(y) \neq 0),$$

divide both sides by $g(y)$ and integrate, to obtain

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx + C.$$

Perform the two integrations and make y the subject of the resulting equation.

3. Integrating factor method (Procedure 4.2)

To solve the linear equation

$$\frac{dy}{dx} = k(x) + l(x)y,$$

subtract $l(x)y$ from both sides and then multiply both sides by the **integrating factor**

$$p(x) = \exp\left[\int (-l(x))\,dx\right].$$

The resulting equation has the form

$$p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y = q(x)$$

and integrates directly to give

$$p(x)y = \int q(x)\,dx + C,$$

which is easily solved for y to give the general solution.

4. Given the general solution of a differential equation involving an arbitrary constant, say C, we can determine the particular solution satisfying a given further condition by substituting the general solution into the further condition. This gives an algebraic equation which we can solve for C. Inserting this value of C into the general solution of the differential equation gives the required particular solution.

5. If you need help with the integrations, consult the Integration section of this Handbook.

Unit 3 Animal populations: their growth and exploitation

Section 1: How to model populations

Mathematical modelling is the use of mathematics in solving real-world problems. **Empiricism** means basing one's model on experiment, without any theory; in the study of population changes it means fitting a simple mathematical relation to the data without seeking to explain why it should have that particular form. If the empirical relation is then applied outside the range covered by the original data, we are **extrapolating**, a notoriously inaccurate procedure.

A useful tool for setting up models which do have some theoretical basis is the **input–output principle**:

increase in population (i.e. accumulation)
= input − output.

The **exponential model** is obtained by assuming that (i) the population $P(t)$ can vary continuously, (ii) migration is negligible and (iii) the numbers of births and of deaths in any short time interval of duration h are respectively $BP(t)h$ and $CP(t)h$, where B and C, the **proportionate birth rate** and **proportionate death rate**, are taken to be constants. The resulting equation is

$$\frac{dP}{dt} = (B - C)P. \qquad (1)$$

The long-term predictions of this model are (i) unbounded growth if $B > C$, (ii) decay to zero if $B < C$ and (iii) constant population if $B = C$. Case (i) in particular is clearly unrealistic, but for some populations these predictions fit the real data reasonably well over limited time intervals.

Section 2: The logistic model

In the **logistic model** we assume that B and C, instead of being constants, depend on the total population in such a way that $B - C$, the **proportionate growth rate**, decreases linearly with the population P. The resulting equation is the **logistic equation**

$$\frac{dP}{dt} = a\left(1 - \frac{P}{M}\right)P, \qquad (2)$$

where a and M are constants. Built into this model are the assumptions that

(i) there is an *equilibrium* population level, M;

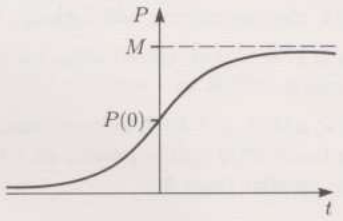
(ii) the proportionate growth rate depends only on the *instantaneous* value of P;

(iii) the proportionate growth rate decreases *linearly* with P, being given by $a(1 - P/M)$.

The solution of the logistic equation is

$$P(t) = \frac{M}{1 + (M/P(0) - 1)e^{-at}},$$

and the graph of this solution (for $0 < P(0) < M$) is called a **logistic curve**.



Two ways of comparing this model with observed values for $P(t)$ are:

(i) estimate M from the property that $P(t)$ is close to M for large t, then plot $\log_e(M/P(t) - 1)$ against t;

(ii) if data are available at equally-spaced instants $t_1, t_2, \ldots$, plot $(P_{n+1} - P_n)/P_n$ against P_{n+1}.

In each case the logistic model predicts that the graph will be a straight line.

Though far from being a 'law of nature', the logistic model agrees quite well with data for some populations, particularly under laboratory conditions; but for wild populations the agreement is more likely to be qualitative than quantitative. In cases where the model does not fit the data well, it is usually possible to see that one or other of the assumptions (i) to (iii) above is not realistic.

Section 3: Fishing

The following procedure, known as the **Schaefer model**, can be used to estimate optimum fishing effort and maximum sustainable yield for a fishery.

(i) Plot yield per unit effort (a measure of population) for each year against the mean fishing effort over the preceding few years.

(ii) Fit a straight line to the points plotted in (i) and assume this to give the relation between the long-term sustainable yield per unit effort and sustained fishing effort.

(iii) Deduce the relation between sustainable yield and sustained fishing effort, and hence find the sustained fishing effort that gives the maximum sustainable yield.

The Schaefer model assumes that

(i) the modified logistic model (3) below applies;

(ii) the historical variations of fishing effort were slow enough for the fish population in each year to be close to the equilibrium level appropriate to the mean level of fishing effort exerted over the previous few years;

(iii) catch is proportional to the product of fish population and fishing effort.

Section 4: Models of exploitation

We can modify the logistic model to include the effect of exploitation by human beings. The new equation is

$$\frac{dP}{dt} = a\left(1 - \frac{P}{M}\right)P - C, \qquad (3)$$

where C is the rate at which members of the population are caught, i.e. it is the **yield**. Two exploitation strategies are considered.

(i) **Constant catch**, i.e. C = constant. Here the conclusions are

  (a) if C exceeds the **maximum sustainable yield** $aM/4$, the population will dwindle to zero;

  (b) even if $C < aM/4$, the population will dwindle to zero if $P(0)$ is too small;

  (c) if $C < aM/4$ and $P(0)$ is large enough, then for long times $P(t)$ will approach an equilibrium value smaller than M.

(ii) **Constant effort**, i.e. C proportional to P, say $C = \lambda f P$, where f (the **effort**) and λ are constants. Here the conclusions are

  (a) if $\lambda f > a$, then the population will dwindle to zero;

  (b) if $\lambda f < a$, then the population is viable whatever the value of $P(0)$;

  (c) in the long run, maximum yield is obtained with a value of $\lambda f = \frac{1}{2}a$.

A more general model, in which (3) is replaced by

$$\frac{dP}{dt} = f(P) - C$$

with $f(P)$ a suitable function of P, gives qualitatively similar conclusions. The qualitative features of the above conclusions are therefore more reliable than the quantitative ones.

An important difference between the two strategies is that if we aim for maximum yield, the constant catch strategy is **unstable**—a small variation in the conditions can cause a large change in the result, and can in this case cause the population to dwindle to zero—whereas the constant effort strategy is **stable**—a small variation in the conditions produces only a small change in the result.

Unit 4 Newtonian mechanics in one dimension

Section 1: The description of motion

1. A **particle** is a material object whose size and internal structure are negligible, so that at any given time it is located at a single point.

2. The motion of a particle along a straight line is described by a **position function** $x(t)$. For each instant t in time, this function specifies the position $x(t)$ occupied by the particle.

3. The **velocity** v of the particle is the derivative of the position function with respect to time, that is,

$$v = \frac{dx}{dt}.$$

4. The **acceleration** a of the particle is the derivative of the velocity function with respect to time. In symbols,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

5. In the special case of **uniform motion**, by which is meant motion with constant velocity v_0, we have

$$x = x_0 + v_0 t,$$
$$v = v_0,$$
$$a = 0,$$

where x_0 is the position at time $t = 0$.

6. In the special case of motion with **constant acceleration** a_0, we have

$$x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2,$$
$$v = v_0 + a_0 t,$$
$$v^2 = v_0^2 + 2a_0(x - x_0),$$

where x_0 and v_0 are the position and velocity of the particle at the instant $t = 0$.

Section 2: Newton's laws of motion

1. Newton's first law states that, in the absence of a force, a particle either stays permanently at rest or moves at constant velocity.

2. The **force** acting on a particle at a given instant is represented by a single number, F, called the force's ***x*-component**. The force has magnitude $|F|$, and is in the direction of increasing x if $F > 0$ or in the direction of decreasing x if $F < 0$. Often 'the force with x-component F' is shortened to 'the force F'.

3. Newton's second law states that if a particle of mass m experiences a net force with x-component F, then its acceleration a is given by $F = ma$.

4. If more than one force acts on a particle, then the **net force** or **total force** F to be used in Newton's second law is the algebraic sum of the individual forces, that is, the sum of their x-components.

Section 3: First model for the motion of a falling object

In this section Newton's second law is used to model the motion of an object falling under the force of gravity, where the effects of air resistance are ignored.

1. Newton's second law $F = ma$ may in principle be integrated by putting

(i) $a = \dfrac{dv}{dt}$ (giving v in terms of t) if $F = F(t)$ or $F = F(v)$;

(ii) $a = v\dfrac{dv}{dx}$ (giving v in terms of x) if $F = F(x)$ or $F = F(v)$.

2. Any object near the Earth's surface is pulled downwards by the **force of gravity**. The magnitude of this force is proportional to the object's inertial mass m, that is,

force of gravity $= mg$ downwards,

where the constant of proportionality g is approximately $9.81\,\mathrm{m\,s^{-2}}$.

Section 4: Second model for the motion of a falling object

In this section Newton's second law is used to model the motion of an object falling under the forces of gravity and air resistance.

1. The motion of any object through the Earth's atmosphere is opposed by the force of **air resistance**, whose magnitude R depends on the object's speed $|v|$, shape and size. For small speeds it is appropriate to use the linear approximation

$$R \simeq k_1|v|,$$

whereas for larger speeds the quadratic approximation

$$R \simeq k_2|v|^2$$

is appropriate. Here k_1 and k_2 are constants which depend on the object's shape and size.

2. The constant speed $|v_\mathrm{T}|$ at which an object can fall is called the **terminal speed**. Using Newton's first law, this occurs when $R = mg$.

Unit 5 Complex numbers

Section 1: Introducing complex numbers

The **natural numbers** are 1, 2, 3, ...; the **integers** are these together with their negatives and zero; **rational numbers** are numbers of the form m/n, where m, n are integers ($n \neq 0$); **real numbers** can be thought of as possible lengths of line segments and are often represented geometrically as points on a **number line**. An **irrational number** is one that is real but not rational.

A **complex number** can be defined as a pair of real numbers (x, y) obeying the following laws of addition and multiplication:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

Like the rational and real number systems, the complex number system is **arithmetically complete** in the sense that the sum, difference, product and quotient of any two numbers in the system (with the single exception of division by zero) is again a number in the system.

If z is a complex number (x, y), then the **real part** of z is $\mathrm{Re}\,z = x$ and the **imaginary part** is $\mathrm{Im}\,z = y$. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. A complex number is (**purely**) **real** if its imaginary part is zero, and (**purely**) **imaginary** if its real part is zero.

The following notation, called the **Cartesian form**, is customary for complex numbers:

(x, y) is written $x + iy$,

$(x, 0)$ is written x,

$(0, y)$ is written iy or yi,

$(0, 1)$ is written i,

$(0, 0)$ is written 0.

Calculations with complex numbers are then easily carried out by following the usual rules of algebra with the additional rule $i^2 = -1$. A list of these rules is given in Subsection 3.6 of this Handbook.

The **complex conjugate** of a complex number $z = x + iy$ is the complex number $\overline{z} = x - iy$, which has the property that $z\overline{z}$ is real and equal to $x^2 + y^2$. To calculate the quotient of two complex numbers, multiply numerator and denominator by the complex conjugate of the denominator.

Section 2: Geometrical representation

The **Argand diagram** is a representation of the complex number system, in which the complex number $x + iy$ is represented by the point in the plane with Cartesian coordinates (x, y). Addition of complex numbers can then be represented by a parallelogram construction, and complex conjugation corresponds to reflection in the x-axis.

Section 3: Polar form

If the point representing a complex number z has polar coordinates $[r, \theta]$ (see Subsection 4.4 of this Handbook), we can write z in the **polar form**

$$z = r(\cos\theta + i\sin\theta).$$

The **modulus** of z is $|z| = r$, and θ is called **an argument** of z. If θ lies in the principal value range $-\pi < \theta \leq \pi$, we call θ **the argument** of z and write $\theta = \mathrm{Arg}\,z$. In terms of x and y, the real and imaginary parts of z, we have

$$|z| = \sqrt{x^2 + y^2},$$
$$\mathrm{Arg}\,z = \begin{cases} \arccos(x/|z|) & \text{if } y \geq 0, \\ -\arccos(x/|z|) & \text{if } y < 0. \end{cases}$$

When two complex numbers are multiplied, the moduli are *multiplied* and their arguments are *added*. In particular, **de Moivre's Theorem** states that, for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Section 4: Euler's formula

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

enables us to write a complex number z in **exponential form** as

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \mathrm{Arg}\,z$, and to define the **exponential function** of z as

$$\exp z = e^z = e^{x+iy} = e^x(\cos y + i\sin y),$$

where $x = \mathrm{Re}\,z$ and $y = \mathrm{Im}\,z$.

Euler's formula also gives the results

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right),$$

which can be used to derive trigonometric identities.

A **sinusoidal function** of time t is one having the form

$$f(t) = A\cos(\omega t + \phi),$$

where the constant A is called the **amplitude**, ω the **angular frequency** and ϕ the **phase**. The function repeats after a time $T = 2\pi/\omega$, called the **period**. The complex number $Ae^{i\phi}$ is called the **phasor** of the sinusoidal function. Sinusoidal functions with the same ω can be added by adding the corresponding phasors.

Section 5: Complex roots

If $a = Re^{i\phi}$ is a complex number and n is a positive integer, the ***n*th roots** of a, written $a^{1/n}$, are the solutions of the equation $z^n = a$. There are precisely n of them, given by

$$a^{1/n} = R^{1/n}\exp\left(i\frac{\phi + 2k\pi}{n}\right) \quad (k = 0, 1, \ldots, n-1).$$

The **fundamental theorem of algebra** states that any polynomial of degree n,

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

(with $a_n \neq 0$), can be factorized into n factors of degree 1,

$$a_n(z - z_1)(z - z_2)\ldots(z - z_n),$$

where $z_1, \ldots, z_n$ are complex numbers. These numbers are called the **roots** of the polynomial and are the solutions of the equation $P(z) = 0$. If the coefficients $a_n, a_{n-1}, \ldots, a_0$ are all real, then the non-real roots (if any) occur in complex conjugate pairs.

For **second-order recurrence relations** of the form

$$u_{r+2} = au_{r+1} + bu_r,$$

if the auxiliary equation $z^2 = az + b$ has non-real solutions $re^{\pm i\theta}$, the general solution is

$$u_n = r^n(A\cos n\theta + B\sin n\theta).$$

Unit 6 Differential equations II

Introductory definitions

A **second-order differential equation** is an equation relating the second derivative of an unknown function to one or more of the following: known functions, the unknown function, and the first derivative of the unknown function. The general solution of such an equation normally involves *two* arbitrary constants. If a second-order differential equation can be written in the form

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = f(x), \tag{1}$$

where p, q, r and f are given functions (and $p(x)$ is not the zero function) then it is **linear**. If in addition f is the zero function, then the equation is **homogeneous**; otherwise it is **inhomogeneous** (or **non-homogeneous**). If a linear second-order differential equation has the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{2}$$

where a, b, c are constants with $a \neq 0$, then it is said to have **constant coefficients**.

Section 1: Homogeneous equations

To solve a homogeneous second-order linear equation with constant coefficients, i.e. one of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{3}$$

apply the following procedure.

(i) Write down the **auxiliary equation**

$$a\lambda^2 + b\lambda + c = 0, \tag{4}$$

and solve it algebraically for λ.

(ii) (a) If (4) has two real distinct solutions λ_1 and λ_2, then the general solution of (3) is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x},$$

where A and B are arbitrary constants.

(b) If (4) has only one real repeated solution λ_1 (i.e. $a\lambda^2 + b\lambda + c$ is a perfect square), then the general solution of (3) is

$$y(x) = (A + Bx)e^{\lambda_1 x}.$$

(c) If (4) has complex conjugate solutions $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then the general solution of (3) is

$$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

If the differential equation is homogeneous but does not necessarily have constant coefficients, the following theorems tell us something about the solutions.

Theorem 1.1: If the functions y_1 and y_2 are solutions of a homogeneous linear differential equation, then any linear combination of y_1 and y_2 is also a solution.

Theorem 1.2: If, in addition, the differential equation is of second order, i.e. of the form

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = 0$$

with the coefficient $p(x)$ never taking the value zero, and if the solutions y_1 and y_2 are linearly independent, then the general solution of the differential equation is $Ay_1 + By_2$, where A and B are arbitrary constants.

By a **linear combination** of two functions y_1 and y_2 with the same domain we mean a function of the form $c_1y_1 + c_2y_2$, where c_1 and c_2 are constants. The functions y_1 and y_2 are said to be **linearly independent** if neither is a constant multiple of the other.

Section 2: Inhomogeneous equations

For the general inhomogeneous Equation (1), we define the **associated homogeneous equation** to be

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = 0. \tag{5}$$

The general solution of (5) is called the **complementary function** of (1), and is denoted here by y_c. The general solution of (1) is (*Theorem 2.1*)

$$y = y_c + y_p,$$

where y_p is any particular solution of (1).

For certain constant-coefficient inhomogeneous equations, particular solutions can usually be found by substituting the function for y_p suggested below into the differential equation and choosing the constants m, n or z (complex) so that the equation is satisfied.

$f(x)$	Try for $y_p(x)$
constant	constant
$kx + l$	$mx + n$
$ke^{\alpha x}$	$me^{\alpha x}$
$k\cos\omega x + l\sin\omega x$ or $\mathrm{Re}(k - il)e^{i\omega x}$	$m\cos\omega x + n\sin\omega x$ or $\mathrm{Re}\, ze^{i\omega x}$

In the above, k, l, α and ω denote given constants.

Exceptionally, the differential equation may have no solution of the suggested form; in that case, multiply the suggested function by x and try again.

The **principle of superposition**: If $u(x)$ is a solution of Equation (1), and if $v(x)$ is a solution of the equation obtained by replacing $f(x)$ in (1) by $g(x)$, then $au(x) + bv(x)$ (with a and b constants) is a solution of the equation obtained by replacing $f(x)$ in (1) by $af(x) + bg(x)$.

Section 3: Initial and boundary conditions

In an **initial condition** problem for Equation (1), the values of y and its derivative are both specified for the same value of x. Such a problem always has a unique solution (so long as $p(x)$ never takes the value zero).

In a **boundary condition** problem, one condition is placed on the value of y (or its derivative) at each of two different values of x. Such a problem may have a unique solution, or no solution, or an infinite number of solutions.

Section 4: Nature of solution functions

For the homogeneous Equation (3), with a and c positive, the solution $y(x)$ may depend on x in various ways, depending on the value of b:

$b < -2\sqrt{ac}$	(for large x) increasing exponential;
$-2\sqrt{ac} < b < 0$	oscillation with increasing amplitude;
$b = 0$	sinusoidal oscillation;
$0 < b < 2\sqrt{ac}$	oscillation with decreasing amplitude;
$2\sqrt{ac} < b$	(for large x) decreasing exponential.

For the inhomogeneous equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = k\cos\omega x + l\sin\omega x,$$

with a, b, c, k, l, ω constants and a, b, c *all* positive, the complementary function decays to zero as x increases, and is called a **transient**. For sufficiently large x, any solution of the equation is close to the unique purely sinusoidal particular solution, which is called the **steady-state** solution.

Section 5: A numerical method

Any second-order differential equation

$$\frac{d^2y}{dx^2} = m\left(x, y, \frac{dy}{dx}\right)$$

can be written as two simultaneous first-order equations

$$\frac{dy}{dx} = z,$$
$$\frac{dz}{dx} = m(x, y, z).$$

Euler's method approximates these by the recurrence relations

$$Y_{r+1} = Y_r + hZ_r,$$
$$Z_{r+1} = Z_r + h\,m(x_r, Y_r, Z_r),$$

where h is the step size, $x_r = x_0 + hr$ and Y_r, Z_r are the approximations to $y(x_r)$ and $z(x_r)$.

Unit 7 Oscillations and energy

Section 1: A home-made oscillating system
Section 2: The perfect spring

In this section a perfect spring is used to model vibrating systems.

1. A **perfect spring** is characterized by two positive constants: its **natural length** and its **stiffness**. It obeys **Hooke's Law**, exerting the following force on any object attached to either of its ends:

(i) when the spring is *extended*, the force is directed towards the centre of the spring and has magnitude (called the **tension**) equal to stiffness × extension;

(ii) when the spring is *compressed*, the force is directed away from the centre of the spring and has magnitude (called the **thrust**) equal to stiffness × compression.

2. If a particle in one-dimensional motion is attached to perfect springs then the equation of motion of the particle can be derived by considering just one configuration in which each spring is designated as being either extended or compressed.

3. The equation of **simple harmonic motion** is

$$\ddot{x} + \omega^2 x = 0,$$

whose general solution can be written either as

$$x = B\cos\omega t + C\sin\omega t$$

or as

$$x = A\cos(\omega t + \phi).$$

The constant ω is called the **angular frequency** and the constant ϕ the **phase angle** of the oscillations. The **period** of the oscillations (time for one complete cycle) is $\tau = 2\pi/\omega$. The **frequency** (number of cycles per second) is $f = 1/\tau = \omega/2\pi$. The **amplitude** of the oscillations is A. The constants A, ϕ and B, C are related via the equations

$$A = \sqrt{B^2 + C^2}$$

and

$$\phi = \begin{cases} \arccos(B/A) & C \le 0, \\ -\arccos(B/A) & C > 0. \end{cases}$$

4. For a particle in simple harmonic motion, the equation $\ddot{x} + \omega^2 x = 0$ describes the motion when the origin of the x-axis is chosen to be at the equilibrium position of the particle. If another origin is chosen then the equation of motion becomes

$$\ddot{x} + \omega^2 x = \omega^2 x_e,$$

where $x = x_e$ is the particle's equilibrium position. For a particle of mass m attached to one end of a perfect spring of stiffness k whose other end is fixed, the angular frequency ω is given by $\omega^2 = k/m$ for either vertical or horizontal motion. Here gravity is taken into account in the vertical case, but other forces are ignored.

Section 3: The conservation of energy

1. The relationship between a force with x-component $F(x)$ and the **potential energy function** $U(x)$ is

$$U(x) = -\int F(x)\,dx,$$

or, equivalently,

$$F(x) = -\frac{dU}{dx}.$$

The point at which the potential energy function is zero is called the **datum**. This can be chosen to be any convenient point.

2. In particular, the **gravitational potential energy** is

$$U = mg \times \text{height},$$

where the height is measured from the chosen datum.

3. The **potential energy of a perfect spring** is

$$U = \tfrac{1}{2} \times \text{stiffness} \times (\text{extension})^2.$$

4. The **law of conservation of mechanical energy** states that if the total force acting on a particle depends only upon the particle's position then the quantity

$$\tfrac{1}{2}mv^2 + U(x) = E$$

remains constant throughout the particle's motion. The quantity

$$T = \tfrac{1}{2}mv^2$$

is known as the **kinetic energy** of the particle, and the constant E as its **total mechanical energy**.

5. The region of motion of a particle with total mechanical energy E satisfies the inequality

$$E - U(x) \geq 0.$$

The end-points of this region, for which

$$E - U(x) = 0,$$

are called the **turning points** of the motion.

Section 4: The fabulous perfect spring

For a system such as a pendulum or a toy car on a curved track, the law of conservation of energy has the form

$$\tfrac{1}{2}m\dot{q}^2 + U(q) = E,$$

where q is the displacement of the pendulum bob along its path or of the car measured along the track. For **small oscillations** about the lowest point of the path, $q = q_0$, the potential energy function can be approximated by the first three terms of its Taylor series. This approximation leads to the prediction of simple harmonic motion of angular frequency $\omega \simeq \sqrt{U''(q_0)/m}$ and period $\tau \simeq 2\pi\sqrt{m/U''(q_0)}$.

Unit 8 Damped and forced vibrations

Section 1: Damped vibrations

1. Damping can be modelled by a damping force of magnitude $r|\dot{x}|$, where r is a positive constant and $\dot{x}$ is the velocity of the particle, and direction opposed to the velocity of the particle. The SI units of the damping constant are $\mathrm{N\,m^{-1}\,s}$ (or $\mathrm{kg\,s^{-1}}$).

2. The equation of motion of a particle of mass m moving under the influence of a perfect spring of stiffness k and linear damping with damping constant r is

$$m\ddot{x} + r\dot{x} + kx = 0,$$

where x is the displacement of the particle from its equilibrium position. Such a system is called a **damped harmonic oscillator**.

3. For **weak damping** ($r^2 < 4mk$) the general solution of the equation of motion is

$$x(t) = Ae^{-\rho t}\cos(\Omega t + \phi),$$

where

$$\rho = \frac{r}{2m} \quad \text{and} \quad \Omega = \frac{\sqrt{4mk - r^2}}{2m}.$$

All solutions of this form may be obtained with the arbitrary constants A and ϕ restricted so that A is non-negative and ϕ lies between $-\pi$ and π.

This motion is oscillatory with period $\tau = 2\pi/\Omega$. The amplitude of the oscillations decreases exponentially, so that

$$x(t+\tau)/x(t) = e^{-\rho\tau}.$$

4. For **critical damping** ($r^2 = 4mk$) the general solution is

$$x(t) = Be^{-\rho t} + Cte^{-\rho t},$$

where $\rho = r/2m$ and B and C are arbitrary constants. This case gives the fastest return to the equilibrium position.

5. For **strong damping** ($r^2 > 4mk$) the general solution is

$$x(t) = Be^{-\rho_1 t} + Ce^{-\rho_2 t},$$

where

$$\rho_1 = \frac{r - \sqrt{r^2 - 4mk}}{2m} \quad \text{and} \quad \rho_2 = \frac{r + \sqrt{r^2 - 4mk}}{2m}.$$

Section 2: Forced vibrations

1. The equation of motion of a damped harmonic oscillator subject to an applied sinusoidal force of amplitude P and angular frequency ω can be written as

$$m\ddot{x} + r\dot{x} + kx = P\cos\omega t.$$

The steady-state **forced vibrations** of this system can be found by using the phasor method.

2. This equation of motion can be rewritten in the form

$$\ddot{x} + 2\alpha\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m}\cos\omega t,$$

where the **undamped angular frequency** ω_0 is defined (as in *Unit* 7) by

$$\omega_0 = \sqrt{\frac{k}{m}},$$

and the **damping ratio** α is defined by

$$\alpha = \frac{r}{2\sqrt{mk}}.$$

3. The steady-state vibration of the forced and damped harmonic oscillator is given by

$$x = A\cos(\omega t + \phi),$$

where

$$A = \frac{P/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}$$

and

$$\phi = -\arccos\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^2\omega^2}}\right).$$

4. For $\alpha < 1/\sqrt{2}$, the amplitude of the forced vibrations exhibits a maximum at a particular value of the forcing angular frequency ω. This effect is known as **resonance**.

Section 3: The perfect dashpot

1. A **perfect dashpot** represents a resistive force which is proportional to the relative velocity between two components of a mechanical system.

2. The dashpot force has magnitude $R = r|\dot{l}|$, where l is the length of the dashpot and r is a positive constant known as the **dashpot constant**. If the dashpot's length is increasing, then the dashpot force is directed towards the centre of the dashpot; if the length is decreasing, then the force is directed away from the centre of the dashpot.

3. The equation of motion of a particle attached to a perfect dashpot can be derived by assuming that the dashpot's length is increasing. This equation of motion applies also to the case where the dashpot is contracting.

Unit 9 Simultaneous linear algebraic equations

In this unit the methods and theory are described for the special case of 3 equations in 3 unknowns, but the descriptions can easily be extended to the case of n equations in n unknowns.

A set of 3 **simultaneous linear algebraic equations** in the 3 unknowns x_1, x_2, x_3 is one that can be written in the form

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= d_1, & E_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= d_2, & E_2 \\ a_3x_1 + b_3x_2 + c_3x_3 &= d_3. & E_3 \end{aligned}$$

The array of numbers

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

is called the **matrix** (of coefficients) of the set of equations, and the array

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

is called the **augmented matrix**.

Section 1: Gaussian elimination

A systematic procedure for solving such sets of equations is **Gaussian elimination**.

Stage 1: Elimination

(i) Eliminate x_1 from all but equation E_1, by subtracting suitable multiples of E_1 from each of the equations below it. In the above example the **multipliers** used in calculating these multiples of E_1 are a_2/a_1 and a_3/a_1 for equations E_2 and E_3 respectively.

(ii) Eliminate x_2 from all but the first two of the new set of equations by subtracting a suitable multiple of the new second equation from the equation below it.

(iii) The new set of equations now has the 'upper triangular' form

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= d_1, & E_1 \\ b_2^*x_2 + c_2^*x_3 &= d_2^*, & E_{2a} \\ c_3^*x_3 &= d_3^*. & E_{3b} \end{aligned}$$

Stage 2: Back-substitution

(i) Solve E_{3b} for x_3.

(ii) Substitute this value of x_3 into E_{2a}, and solve the resulting equation for x_2.

(iii) Similarly, substitute these values of x_3 and x_2 into E_1, and hence find x_1.

Section 2: Types of solution

1. A set of linear simultaneous equations will have:

(i) a unique solution, or

(ii) no solution, or

(iii) an infinite number of solutions.

To find out which case applies, look at the augmented matrix of the set of equations produced by stage 1 of Gaussian elimination. [In the table below, a matrix entry which must be non-zero is denoted by ✓, while a matrix entry which could be zero or non-zero is denoted by a star (∗).]

Characteristics of matrix obtained after elimination	Type of solution
$\left[\begin{array}{ccc\|c} ✓ & * & * & * \\ 0 & ✓ & * & * \\ 0 & 0 & ✓ & * \end{array}\right]$ No zero row of new coefficient matrix	Unique solution (case (i))
$\left[\begin{array}{ccc\|c} ✓ & * & * & * \\ 0 & ✓ & * & * \\ 0 & 0 & 0 & ✓ \end{array}\right]$ A zero row of new coefficient matrix, but not of new augmented matrix (i.e. Equation E_{3b} has form 0 = non-zero number)	No solution (equations inconsistent; case (ii))
$\left[\begin{array}{ccc\|c} ✓ & * & * & * \\ 0 & ✓ & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ A zero row of the new augmented matrix (i.e. Equation E_{3b} has form 0 = 0)	An infinite number of solutions (case (iii))

In case (iii) we can obtain the solution set by taking $x_3 = k$, where k is an arbitrary number, then back-substituting in Equations E_{2a} and E_1 as usual.

2. The rows $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ of a matrix are said to be **linearly dependent** if some **linear combination** of these rows is zero, i.e. if constants $k_1, k_2, \ldots, k_n$ (not all zero) exist such that

$$k_1\mathbf{R}_1 + k_2\mathbf{R}_2 + \cdots + k_n\mathbf{R}_n = \mathbf{0},$$

where $\mathbf{0}$ here means a row of zeros.

If the rows of the matrix $\mathbf{A}$ of coefficients in the given set of equations are linearly independent, then the solution is unique (case (i)). If the rows of $\mathbf{A}$ are linearly dependent but those of the augmented matrix $\mathbf{A}|\mathbf{b}$ are linearly independent, then there is no solution (case (ii)). If the rows of $\mathbf{A}|\mathbf{b}$ (and hence also those of $\mathbf{A}$) are linearly dependent, then there is an infinite number of solutions (case (iii)).

Section 3: Some possible difficulties

1. (i) The kth **pivot** (or pivot element) is the one by which we divide to obtain the multipliers in the kth step of the elimination.

(ii) The **pivot row** is the row in which the current pivot lies.

(iii) **Essential row interchange** is the process of interchanging the rows of a matrix when the obvious pivot is zero. This can be done by interchanging the current pivot row with whichever of the rows beneath it will provide the pivot with the largest modulus.

(iv) **Partial pivoting** is the process described in the preceding sentence with the difference that it is done even if the obvious pivot is not zero, provided that it will replace the obvious pivot by a pivot with a larger modulus. It ensures that no multiplier has modulus exceeding 1.

2. The section shows some of the difficulties presented by simultaneous equations and their solution.

(i) **Ill-conditioning**. A set of equations is said to be ill-conditioned if a 'small' change in the data (the coefficients, or the right-hand sides of the equations) produces 'large' changes in the solution. Ill-conditioning is a property of the equations themselves, not of the method used to solve them. One way to detect ill-conditioning is to vary the coefficients slightly and see if there is much change in the solution. A possible cure for ill-conditioning is to replace one of the equations by an equation containing some new independent information, and hence obtain a set of equations which are not ill-conditioned.

(ii) **Induced instability**. This occurs when we start with a set of equations which are not ill-conditioned and yet the method used to solve the equations induces ill-conditioning in the problem during the process of solution. It is usually heralded by the appearance of a very large multiplier. Partial pivoting is used as a way of trying to avoid this (and to avoid the consequent build-up of error).

Section 4: The computer package

Section 5: Special cases

If the right-hand sides of the equations are all zero, the equations are said to be **homogeneous**. If the number of unknowns is the same as the number of equations, and the equations are linearly independent, then the only solution is the **trivial solution**, in which all unknowns equal zero. If, however, the equations are linearly dependent, then there is an infinite number of solutions.

A set of m equations in n unknowns has an infinite number of solutions if $m < n$ (unless the equations are inconsistent) and no solutions at all if $m > n$ (unless at least $m - n$ of the equations are linearly dependent on the others).

Unit 12 Heat transfer

Section 1: The nature of heat transfer

1. Heat transfer means energy transfer by heating and/or cooling. This kind of energy transfer requires the existence of a temperature difference, and the energy is transferred from regions or bodies of higher temperature to those of lower temperature. There are three modes of heat transfer: **conduction** within a (normally stationary) material, **radiation** (or electromagnetic radiation) which does not require any material, and **convection** by motion of a fluid. In **free convection** the fluid motions, or **convection currents**, are driven by density changes caused by the temperature difference itself; in **forced convection**, they are driven by some other agent.

2. Part of the energy of a body depends on its temperature. This part is known as its **internal energy**, and the change of internal energy due to a change from a temperature θ_1 to a temperature θ_2 is $mc(\theta_2 - \theta_1)$, where

m = mass of body,

c = **specific energy capacity** or **specific heat** of the material.

Power is the rate at which energy (of some particular kind) is produced.

Section 2: Steady-state conduction in one dimension

1. Fourier's Law for steady-state conduction in one dimension is

$$q = -\kappa A \frac{d\theta}{dx},$$

where

q = heat transfer rate in the positive x direction,

κ = **thermal conductivity** of the material,

A = area at right angles to the direction of heat transfer,

θ = temperature,

x = distance measured in direction of temperature variation.

$\dfrac{d\theta}{dx}$ is known as the **temperature gradient**.

Steady-state means that the temperature at any given point does not change with time; **one-dimensional** means that it depends on only one of the three possible space coordinates (in a suitable coordinate system).

2. For steady-state conduction through a uniform slab of constant cross-sectional area A, Fourier's Law reduces to

$$q = \kappa A \frac{\theta_1 - \theta_2}{b},$$

where $\theta_1 - \theta_2$ (for $\theta_1 > \theta_2$) is the temperature drop across thickness b of the material.

For steady-state conduction through a pipe of circular cross-section with internal radius r_1, external radius r_2 and length l, Fourier's Law leads to

$$q = \frac{2\pi\kappa l(\theta_1 - \theta_2)}{\log_e(r_2/r_1)},$$

where $\theta_1 - \theta_2$ (for $\theta_1 > \theta_2$) is the temperature drop across the pipe wall.

Section 3: Walls and radiation

1. Heat transfer through walls and roofs is often represented by steady-state one-dimensional models. The ***U* value** of a wall is defined as follows:

$$U \text{ value} = \frac{\text{rate of heat transfer per unit area}}{\text{overall temperature difference}}.$$

New techniques and components are being developed to satisfy the demand for very low U values in buildings.

2. The thermal energy emitted by radiation from a body depends on the fourth power of the **absolute temperature** of the body, i.e.

$$\text{thermal energy emitted by radiation} = \mu\theta_A^4,$$

where

μ is a constant,

$\theta_A = 273.2 + \theta$ = absolute temperature (in kelvins),

θ = temperature in °C.

Section 4: Convection and insulation

1. Convective heat transfer in a fluid can be modelled using the following assumptions:

(i) the temperature in the bulk fluid is uniform, say θ_f;

(ii) if the fluid meets a plane solid surface with area A and temperature θ_s, then the rate of heat transfer from surface to fluid is $hA(\theta_s - \theta_f)$, where h is the **convective heat transfer coefficient**.

2. The equation for heat transfer through a wall or window with area A is

$$q = AU(\theta_{\text{in}} - \theta_{\text{out}}),$$

where θ_{in} and θ_{out} (for $\theta_{\text{in}} > \theta_{\text{out}}$) are the bulk air temperatures inside and outside the building. Overall U values which take both conduction and convection into account can be calculated by adding formulas for adjacent temperature differences.

In particular, for a solid wall

$$U = \left\{ \frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}} \right\}^{-1},$$

where b and κ are the thickness and thermal conductivity of the wall and h_{in} and h_{out} are the convective heat transfer coefficients at the inside and outside surfaces of the wall respectively.

3. Convective heat transfer from a surface such as the wall of a pipe can be increased by means of fins. The heat transfer in the fin can be modelled by assuming that the temperature in the fin depends only on its distance x from the pipe wall, dividing the fin into elements, and then considering convection from each element of the fin to the surrounding air and conduction from each element to its neighbours. This leads to the differential equation

$$\frac{d^2\theta}{dx^2} = \frac{2h}{\kappa b}(\theta - \theta_a),$$

where h is the convective heat transfer coefficient, κ is the thermal conductivity of the material of the fin, b is the fin's thickness and θ_a is the temperature of the air.

4. The heat transfer properties of a layer of fluid (such as the air space in double glazing) are complicated, and are best modelled using a **combined heat transfer coefficient** h_c, which is defined analogously to a U value but depends on the temperature difference across the layer as well as on its thickness.

Unit 14 Vector algebra

Section 1: Vectors: scaling and addition

1. A (non-zero) **vector** is a mathematical object consisting of two parts:

(i) a positive real number called the **modulus** or **magnitude** of the vector, and

(ii) a direction in space (or in the plane).

The modulus of a vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$ or sometimes by v. We also define the **zero vector** to have modulus zero; no direction is defined for it. The zero vector is denoted by $\mathbf{0}$. In the context of vectors, we use the word **scalar** to denote a real number.

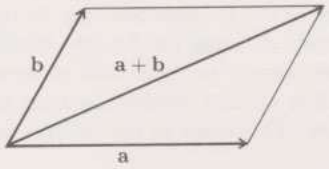
2. Physical quantities that can be modelled by vectors or by scalars are called **vector quantities** or **scalar quantities** respectively. Examples of vector quantities include displacements (i.e. changes in position), velocities and forces. The displacement from a point P to a point Q is represented by the **displacement vector** $\overrightarrow{PQ}$.

3. **Equal** vectors have the same magnitude and direction. **Scaling** of a vector is **multiplying** it by a positive number; this leaves the direction unaltered and multiplies the magnitude by that number. Multiplying any vector by the number zero gives the zero vector. Multiplying a vector $\mathbf{v}$ by a negative number $-m$ gives a vector $-m\mathbf{v}$ with magnitude $m|\mathbf{v}|$ and direction opposite to that of $\mathbf{v}$. This new direction has the same orientation (i.e. in two dimensions, the same slope) as that of $\mathbf{v}$ but its sense is opposite to that of $\mathbf{v}$. In particular the vector $-\mathbf{v}$ has the same magnitude as $\mathbf{v}$ and the opposite direction.

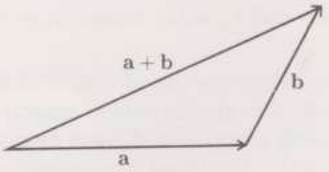
4. A **unit vector** is one whose magnitude is 1. The vector $\frac{1}{|\mathbf{a}|}\mathbf{a}$ is a unit vector with the same direction as $\mathbf{a}$.

Unit vectors along the Cartesian axes are called **Cartesian unit vectors**, and are denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

5. Vector addition is defined by a parallelogram rule



or alternatively by a triangle rule



Vector subtraction is defined by

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

6. Vector addition is commutative and associative (see Subsection 3.6 of this Handbook), and distributive with respect to multiplication by a scalar:

$$(m_1 + m_2)\mathbf{a} = m_1\mathbf{a} + m_2\mathbf{a},$$

$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$$

Multiplication by a scalar has the 'associative' property that

$$m_1(m_2\mathbf{a}) = (m_1 m_2)\mathbf{a}.$$

The zero vector has the properties $0\mathbf{a} = \mathbf{0}$ and $\mathbf{0} + \mathbf{a} = \mathbf{a}$.

Section 2: Components of a vector

1. Given a Cartesian coordinate system, any two-dimensional vector $\mathbf{a}$ can be written uniquely in the **component form**

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the positive x- and y-axes respectively. The vectors $a_1\mathbf{i}$ and $a_2\mathbf{j}$ are called the (**Cartesian**) **component vectors** of $\mathbf{a}$, and the numbers a_1 and a_2 are called the (**Cartesian**) **components** of $\mathbf{a}$ in the directions of $\mathbf{i}$ and $\mathbf{j}$. The magnitude of $\mathbf{a}$ is $\sqrt{a_1^2 + a_2^2}$. To add two vectors, we add the respective components; to multiply a vector by a scalar, we multiply the respective components by that scalar.

2. For three-dimensional vectors the representation

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

has analogous properties, where $\mathbf{k}$ is a unit vector along the positive z-axis, chosen at right angles to the x- and y-axes. A **right-handed coordinate system** is one satisfying the **right-hand rule**: if the thumb of your right hand points along $\mathbf{i}$ and the forefinger along $\mathbf{j}$, then the middle finger can comfortably point along $\mathbf{k}$. A right-handed coordinate system also satisfies the **right-hand screw rule** that a screw rotated from $\mathbf{i}$ to $\mathbf{j}$ will advance in the direction of $\mathbf{k}$.

3. If a point P has coordinates x, y, z relative to a given Cartesian coordinate system with origin O, the displacement vector $\overrightarrow{OP}$ is called the **position vector** of P, and is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Section 3: Products of vectors

1. The **dot product** or **scalar product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ (for $0 \leq \theta \leq \pi$) is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$.

In particular we have

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2.$$

The angle θ between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

2. The **projection** of a vector $\mathbf{b}$ onto a given direction is defined to be $|\mathbf{b}| \cos\theta$, where θ is the angle between the direction of $\mathbf{b}$ and the given direction; it is equal to $\mathbf{a} \cdot \mathbf{b}/|\mathbf{a}|$, where $\mathbf{a}$ is any vector whose direction is the given direction (it is often convenient to make $\mathbf{a}$ a unit vector). Hence

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|$ times the projection of $\mathbf{b}$ onto the direction of $\mathbf{a}$

$= |\mathbf{b}|$ times the projection of $\mathbf{a}$ onto the direction of $\mathbf{b}$.

3. The dot product is commutative,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$

and distributive,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c},$$

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c},$$

but it is non-associative and there is no inverse. It combines with scaling according to the rule

$$(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b}).$$

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors.

4. Dot products of Cartesian unit vectors are

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

If two vectors $\mathbf{a}$ and $\mathbf{b}$ have Cartesian components a_1, a_2, a_3 and b_1, b_2, b_3 respectively, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The formula $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$ is a particular case of this result.

5. The **cross product** or **vector product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector defined by

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\,\mathbf{c},$$

where θ (for $0 \leq \theta \leq \pi$) is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{c}$ is a unit vector at right angles to both $\mathbf{a}$ and $\mathbf{b}$. The sense of $\mathbf{c}$ is given by the **right-hand screw rule**, that a screw rotated from $\mathbf{a}$ to $\mathbf{b}$ advances along the direction of $\mathbf{c}$.

6. The cross product is distributive,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c},$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c},$$

but is non-commutative, obeying instead the law

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}).$$

It is non-associative, and there is no inverse. It combines with scaling according to the rule

$$(m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (m\mathbf{b}).$$

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ have the same **orientation** (i.e. they are parallel or anti-parallel). In particular, we have $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

7. Cross products of the Cartesian unit vectors are

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

If two vectors $\mathbf{a}$ and $\mathbf{b}$ have Cartesian components a_1, a_2, a_3 and b_1, b_2, b_3 respectively, then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

8. The **triple scalar product** of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

9. Given a plane, any vector $\mathbf{n}$ perpendicular to it is called a **normal vector**, and the equation of the plane can be written

$$(x - a)n_1 + (y - b)n_2 + (z - c)n_3 = 0,$$

where (a, b, c) is a point in the plane and $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$. If $\mathbf{a}$ and $\mathbf{b}$ are vectors lying in this plane, with different orientations, then $\mathbf{a} \times \mathbf{b}$ is a normal vector.

Section 4: Differentiating vectors

1. A **vector function** is a function whose domain is the set of all real numbers and whose codomain is the set of all vectors: to each real number t the vector function $\mathbf{f}$ assigns a vector $\mathbf{f}(t)$. Alternatively, the formula for $\mathbf{f}(t)$ may itself be described as a (vector) function of t. The Cartesian components of a vector function are real functions of the usual kind:

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}.$$

The magnitude and direction of $\mathbf{f}(t)$ are also functions of t. A vector whose Cartesian components are constants (and hence whose magnitude and direction are constants) is called a **constant vector**.

2. As t varies, the point whose position vector (relative to a given coordinate system) is $\mathbf{f}(t)$ traces out a curve in three dimensions, which helps in visualizing the vector function $\mathbf{f}$.

3. The **derivative** of a vector function $\mathbf{f}$ is

$$\frac{d\mathbf{f}(t)}{dt} = \lim_{h \to 0} \left(\frac{\mathbf{f}(t + h) - \mathbf{f}(t)}{h} \right).$$

If $\mathbf{f}(t)$ has the Cartesian component form

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k},$$

then

$$\frac{d\mathbf{f}(t)}{dt} = \frac{df_1(t)}{dt}\mathbf{i} + \frac{df_2(t)}{dt}\mathbf{j} + \frac{df_3(t)}{dt}\mathbf{k}.$$

The rules for differentiating sums and products are

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt},$$
$$\frac{d}{dt}(m\mathbf{u}) = \frac{dm}{dt}\mathbf{u} + m\frac{d\mathbf{u}}{dt},$$
$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt},$$
$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \left(\frac{d\mathbf{u}}{dt} \times \mathbf{v}\right) + \left(\mathbf{u} \times \frac{d\mathbf{v}}{dt}\right).$$

4. The tangent at any given point on the curve representing a vector function $\mathbf{f}$ is in the direction of the vector $\dfrac{d\mathbf{f}(t)}{dt}$ at that point. If $\mathbf{r}(t)$ is the position vector of a particle at time t, then its velocity vector is $\dfrac{d\mathbf{r}(t)}{dt}$. If a vector function $\mathbf{g}(t)$ has constant magnitude, then $\mathbf{g} \cdot \dfrac{d\mathbf{g}}{dt} = 0$.

Unit 15 Newtonian mechanics in three dimensions

Section 1: The ingredients of Newton's second law

1. In three-dimensional Newtonian mechanics the position of a particle is represented by the **position vector**

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

where x, y and z are the coordinates of the particle, and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the axes of a static right-handed Cartesian coordinate system that is calibrated in metres.

2. The **velocity vector** $\mathbf{v}$ is found by differentiating the position vector with respect to t, the time measured in seconds:

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}.$$

The magnitude of the velocity vector is the **speed** of the particle. The direction of the velocity vector is the **direction of motion** of the particle.

3. The **acceleration vector** $\mathbf{a}$ is found by differentiating the velocity vector with respect to t:

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}.$$

This describes the rate of change of velocity both in magnitude and direction. The acceleration will be non-zero if *either* the speed *or* the direction of motion changes.

4. Each individual force acting on a particle is represented by a **force vector**. If the direction of the force makes angles α, β and γ with the positive x-, y- and z-axes, and if the magnitude of the force is $|\mathbf{F}|$, then the force vector is

$$\mathbf{F} = |\mathbf{F}|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}).$$

The cosines that appear in this equation can sometimes be simplified by using the trigonometric identities

$$\cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi, \quad \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi,$$
$$\cos(\pi - \phi) = -\cos\phi.$$

5. The **gravitational force vector** for a particle of mass m is $mg\mathbf{e}$, where $\mathbf{e}$ is a unit vector pointing vertically downwards and $g \simeq 9.81\,\mathrm{m\,s^{-2}}$.

6. If a particle experiences simultaneously a number of individual forces, represented by the vectors $\mathbf{F}_1, \mathbf{F}_2, \ldots$, it behaves just as if it were acted upon by the single force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots.$$

This formula is known as the **law of addition of forces**, and the vector sum $\mathbf{F}$ is known as the **total force vector**. The components of the total force vector are denoted by F_x, F_y and F_z.

Section 2: Newton's second law

1. **Newton's second law**, in its vector formulation, states that, at each instant of time, the motion of a particle is governed by the equation

$$m\ddot{\mathbf{r}} = \mathbf{F}, \quad \text{i.e. } m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}$$

or, in terms of components,

$$m\ddot{x} = F_x, \quad m\ddot{y} = F_y, \quad m\ddot{z} = F_z,$$

where

m is the mass of the particle,

$\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$ is the acceleration vector,

$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ is the total force vector.

2. A particle that moves in a circle with constant speed is said to perform **uniform circular motion**, and its acceleration is directed towards the centre of the circle. For example, a particle of mass m for which

$$\mathbf{r}(t) = l(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}),$$

where ω is a constant, performs uniform circular motion at speed $l\omega$ round a circle of radius l. Its acceleration is $-\omega^2\mathbf{r}(t)$ and the force on it is $-m\omega^2\mathbf{r}(t)$, which has magnitude $m\omega^2 l$ and is directed from the particle towards the centre of the circle.

Section 3: Algebraic equations and sloping tables

1. A **static** particle, i.e. one which is at rest over some interval of time, experiences zero total force.

2. A particle in contact with a solid surface experiences a normal reaction force, $\mathbf{F}_N$, and a force due to friction, $\mathbf{F}_f$.

3. The **normal reaction rule**: $\mathbf{F}_N$ is perpendicular to the surface and $|\mathbf{F}_N|$ is just sufficient to prevent the particle from sinking into the surface. For the particle to remain in contact with the surface, $|\mathbf{F}_N| \geq 0$.

4. The **force of friction**, or **frictional reaction**, $\mathbf{F}_f$, is parallel to the surface, in the direction that opposes motion. For a static particle, $|\mathbf{F}_f|$ is just sufficient to prevent sliding over the surface, but the particle can remain static only so long as $|\mathbf{F}_f| < \mu|\mathbf{F}_N|$, where μ is the **coefficient of static friction**. For a moving particle, $|\mathbf{F}_f| = \mu'|\mathbf{F}_N|$, where μ' is the **coefficient of kinetic friction**.

Section 4: Uncoupled differential equations and shot-putters

1. We can model a projectile, such as the shot, as a particle that experiences only the downward force of gravity. The **path** or **trajectory** of a moving point is the set of points it passes through. The path of the shot is a **parabola**, that is, the graph of a quadratic function (with the y-axis vertical).

2. If the shot-putter launches from zero height, the maximum range is

$$\frac{v_{\max}^2}{g},$$

where $v_{\max}$ is the greatest speed with which he can release the shot. The maximum range is achieved for an angle of release of $\frac{1}{4}\pi$ radians.

3. If the shot-putter launches from a height h, the maximum range is

$$L\sqrt{1 + 2h/L},$$

where $L = v_{\max}^2/g$, and this is achieved for an angle of release of

$$\arctan\left(\frac{1}{\sqrt{1 + 2h/L}}\right).$$

Section 5: Coupled differential equations and pendulums

1. This section discusses a particle of mass m suspended from a fixed point by a **light taut inextensible** string, where 'light' means 'of negligible mass', 'inextensible' means that l, the length of the string, is constant, and 'taut' means that the distance between the two ends of the string is l.

2. If the particle moves in a horizontal circle, then its speed is constant, the tension in the string is constant, and the equations of motion for x, y and z are uncoupled (i.e. the equation for $\ddot{x}$ does not involve y or z, and so on).

3. If the particle moves in a vertical circle, then its speed is *not* constant, the tension in the string is *not* constant and the equations of motion are coupled. In this case the angle θ between the string and the downward vertical satisfies the equation of motion

$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

This is consistent with the energy equation

$$\tfrac{1}{2}m(l\dot{\theta})^2 + mgl(1 - \cos\theta) = E = \text{constant}.$$

The tension in the string is

$$\begin{aligned} T &= ml\dot{\theta}^2 + mg\cos\theta \\ &= mg(3\cos\theta - 2) + \frac{2E}{l} \end{aligned}$$

so long as the right-hand side of this equation is positive. As soon as $T = 0$ the string goes slack, and from then onwards a different description of the motion must be used.

4. The sum of the kinetic and gravitational potential energies of a particle is conserved provided the particle experiences only the force of gravity and forces (like the tension in a string, the normal reaction force or the magnetic force on a charged particle) that act at right angles to its velocity.

Unit 17 The dynamics of many-particle systems

Section 1: Two-particle string–pulley systems

Section 2: Systems of particles

1. Newton's third law states that the inter-particle forces acting mutually between a pair of particles are equal in magnitude and opposite in direction. If the particles are labelled i and j, then this law may be expressed in terms of vectors by the equation

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji},$$

where $\mathbf{F}_{ij}$ is the force on Particle i due to the effect of Particle j, and vice versa.

2. For a system of n particles with masses $m_1, m_2, \ldots, m_n$ and respective position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, the **centre of mass** has position vector

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n) = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i,$$

where

$$M = m_1 + m_2 + \cdots + m_n = \sum_{i=1}^{n} m_i$$

is the **total mass** of the system. If the particles are acted upon by external forces $\mathbf{F}_1^{\text{ext}}, \mathbf{F}_2^{\text{ext}}, \ldots, \mathbf{F}_n^{\text{ext}}$ respectively, and if the inter-particle forces satisfy Newton's third law, then the equation of motion of the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}},$$

where

$$\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \cdots + \mathbf{F}_n^{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}$$

is the **total external force** acting on the system.

3. In the absence of external forces, the equation of motion of the centre of mass reduces to $\ddot{\mathbf{R}} = \mathbf{0}$, indicating a uniform motion

$$\mathbf{R}(t) = \mathbf{V}_0 t + \mathbf{R}_0,$$

where the constant vectors $\mathbf{V}_0$ and $\mathbf{R}_0$ are respectively the velocity of the centre of mass and its initial position (at time $t = 0$).

4. The centre of mass of a symmetric homogeneous object is located at the geometric centre of the object. The centre of mass of a collection of objects can be found by treating each object as if it were a single particle situated at its own centre of mass.

Section 3: Linear momentum and energy

1. The **linear momentum p** of a particle with mass m and position vector $\mathbf{r}$ is defined by

$$\mathbf{p} = m\dot{\mathbf{r}}.$$

In terms of linear momentum, **Newton's second law** can be expressed as

$$\dot{\mathbf{p}} = \mathbf{F},$$

where $\mathbf{F}$ is the total force acting upon the particle. In the absence of any force on the particle, its linear momentum is conserved (that is, $\mathbf{p}$ is constant). If the mass m of the particle is constant, then Newton's second law reduces to

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

2. For a many-particle system, the **total linear momentum P** is the sum of the momenta of the individual particles. It is related to the total mass M of the system and the centre of mass $\mathbf{R}$ by the equation

$$\mathbf{P} = M\dot{\mathbf{R}}.$$

Newton's second law leads to the result

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}},$$

where $\mathbf{F}^{\text{ext}}$ is the total external force acting on the system. In the absence of external forces, the total linear momentum is conserved (that is, $\mathbf{P}$ is constant). If the total mass M of the system is constant, then the last equation reduces to

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}.$$

3. For a system of particles involved in a collision in the absence of external forces, momentum conservation implies that $\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}}$. In the two-particle case, this is equivalent to

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2,$$

where the particles have masses m_1, m_2, initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$, respectively.

4. If a two-particle collision is **elastic**, then the kinetic energy equation

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2$$

is also satisfied. Otherwise the collision is **inelastic**.

5. Newton's law of restitution for a two-particle collision is

$$(\mathbf{v}_2 - \mathbf{v}_1)\cdot\mathbf{l} = -e(\mathbf{u}_2 - \mathbf{u}_1)\cdot\mathbf{l},$$

where $\mathbf{l}$ is a unit vector in the direction of individual particle momentum change, and the constant e is the **coefficient of restitution** for the objects represented by the particles. For an elastic collision, the coefficient of restitution is $e = 1$. If the collision is one-dimensional, then this law reduces to

$$v_2 - v_1 = -e(u_2 - u_1).$$

Section 4: Newton's third law

Section 5: Rocket motion

1. The velocity $\mathbf{v}$ of a rocket satisfies the **rocket equation**

$$m\dot{\mathbf{v}} - \dot{m}\mathbf{u} = \mathbf{F}^{\text{ext}},$$

where the rocket (with unspent fuel) has mass m, is acted upon by an external force $\mathbf{F}^{\text{ext}}$ and emits exhaust gases with relative velocity $\mathbf{u}$.

2. In the absence of any external force, the rocket equation reduces to

$$\dot{\mathbf{v}} = \frac{\dot{m}}{m}\mathbf{u}.$$

Unit 18 Polynomial approximations

Section 1: Basic theory

1. A **polynomial** of **degree** n is a function that can be expressed in the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$. The constant numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are called the **coefficients**.

2. To evaluate the above polynomial for a given value of x by **nested multiplication**, use the recurrence relation

$$u_{r+1} = xu_r + a_{n-r-1} \quad \text{for } r = 0, 1, 2, \ldots, n-1,$$

with $u_0 = a_n$. Then u_n is the value of the polynomial.

3. For a given function $f(x)$ and a given number α in its domain, the ***n*th Taylor polynomial** for f about α is

$$f(\alpha) + (x-\alpha)f'(\alpha) + \frac{1}{2!}(x-\alpha)^2 f''(\alpha)$$
$$+ \cdots + \frac{1}{n!}(x-\alpha)^n f^{(n)}(\alpha) = \sum_{k=0}^{n} \frac{1}{k!}(x-\alpha)^k f^{(k)}(\alpha).$$

4. Taylor's Theorem. Let the function f and its first $n+1$ derivatives exist and be continuous at all points between and including two given numbers α and x. Then we have

$$f(x) = p(x) + r(x),$$

where

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-\alpha)^k f^{(k)}(\alpha)$$

is the nth Taylor polynomial for f about α, and $r(x)$, the **remainder**, satisfies

$$r(x) = \frac{1}{(n+1)!}(x-\alpha)^{n+1} f^{(n+1)}(c_x)$$

for some number c_x between α and x. Taylor's Theorem with $n = 0$ is also called the **Mean Value Theorem**.

5. The **error function** for the approximation of a given function $f(x)$ by a polynomial $p(x)$ is

$$\varepsilon(x) = p(x) - f(x).$$

In the case of Taylor approximations it is related to the remainder by

$$\varepsilon(x) = -r(x).$$

6. Suppose f is a given function and a, b, M are numbers such that

$$|f^{(n+1)}(c)| \leq M$$

for all c satisfying $a \leq c \leq b$. Then, for any α satisfying $a \leq \alpha \leq b$, the error in the nth Taylor polynomial approximation to f about α satisfies the **error bound** formula

$$|\varepsilon(x)| \leq \frac{1}{(n+1)!}|x-\alpha|^{n+1} M \quad \text{for } a \leq x \leq b.$$

Section 2: Two applications of Taylor polynomials

1. A **root** of the equation $f(x) = 0$ means a solution of that equation.

2. To calculate a root of the equation $f(x) = 0$ by the **Newton–Raphson method**, use the recurrence relation

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}.$$

If the initial guess x_0 is not too far out, the sequence $x_0, x_1, x_2, \ldots$ will converge to the root (i.e. for sufficiently large r, the value of x_r is arbitrarily close to the root). The error in x_{r+1} is proportional to the square of the error in x_r; so if the sequence converges at all it converges rapidly.

3. The **Taylor series method of order *n*** for the solution of the first-order differential equation

$$y' = m(x, y),$$

where the value of $y(x_0)$ is given, involves the following steps.

(i) Differentiate the expression $m(x, y)$ with respect to x (remembering that y is a function of x) $n-1$ times to obtain expressions for

$$\frac{dm}{dx}(x, y), \quad \frac{d^2m}{dx^2}(x, y), \quad \ldots, \quad \frac{d^{n-1}m}{dx^{n-1}}(x, y).$$

(ii) Choose a step size h.

(iii) Calculate $Y_1, Y_2, \ldots$ from the recurrence relation

$$Y_{r+1} = Y_r + hY_r' + \frac{1}{2!}h^2 Y_r'' + \cdots + \frac{1}{n!}h^n Y_r^{(n)},$$

where

$$x_r = x_0 + rh,$$
$$Y_r' = m(x_r, Y_r),$$
$$Y_r'' = \frac{dm}{dx}(x_r, Y_r),$$
$$\vdots$$
$$Y_r^{(n)} = \frac{d^{n-1}m}{dx^{n-1}}(x_r, Y_r),$$

with the initial condition

$$Y_0 = y(x_0).$$

(iv) Y_r is an approximation to $y(x_r)$.

Section 3: The catenary

1. Hyperbolic cosines and **sines** are functions defined by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

They have the properties

$$\cosh^2 x - \sinh^2 x = 1,$$
$$\frac{d}{dx}\cosh x = \sinh x,$$
$$\frac{d}{dx}\sinh x = \cosh x,$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots,$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$$

These functions are related to the hyperbola $x^2 - y^2 = 1$, in the same way that the ordinary cosine and sine functions are related to the circle.

2. The length L of a curve with equation $y = y(x)$, between two points on it with x-coordinates a and b, is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

3. The curve with equation

$$y = \frac{1}{a}\cosh ax,$$

where a is any positive constant, is called a **catenary**. It models the shape of a rope suspended between two points.

Section 4: Interpolation and integration

1. If the values of a function $f(x)$ are known at points $x_0, x_1, \ldots, x_n$, arranged in increasing order but not necessarily equally spaced, then the use of some other function $g(x)$, which takes the same values as $f(x)$ for $x = x_0, x_1, \ldots, x_n$, to approximate $f(x)$ for values of x between x_0 and x_n, is called **interpolation**; the corresponding procedure for values of x less than x_0 or greater than x_n is called **extrapolation**.

2. There is a unique **interpolating polynomial** of degree $\leq n$ which takes the same values as $f(x)$ for $x = x_0, x_1, \ldots, x_n$. Its $n+1$ coefficients, say $a_0, a_1, \ldots, a_n$, can be found by solving the $n+1$ simultaneous linear equations

$$a_0 + a_1 x_i + \cdots + a_n x_i^n = f(x_i) \quad (i = 0, 1, \ldots, n).$$

3. Provided $x_0 \leq x \leq x_n$, the error function $\varepsilon(x)$ for the approximation of $f(x)$ by the interpolating polynomial defined above satisfies

$$\varepsilon(x) = -\frac{1}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n) f^{(n+1)}(c_x),$$

where c_x is some number satisfying $x_0 \leq c_x \leq x_n$. If M is a number such that

$$|f^{(n+1)}(c)| \leq M$$

for all c between x_0 and x_n, then $\varepsilon(x)$ satisfies the error bound formula

$$|\varepsilon(x)| \leq \frac{1}{(n+1)!}|(x - x_0)(x - x_1)\ldots(x - x_n)|M.$$

4. Euler's integration formula, based on approximating the area under a segment of a curve by a rectangle, is

$$\int_a^b f(x)\, dx \simeq h(f(x_0) + f(x_1) + \cdots + f(x_{n-1})),$$

where

$$h = (b - a)/n$$

and $x_0, x_1, \ldots, x_{n-1}$ are equally-spaced points defined by

$$x_r = a + rh \quad (r = 0, 1, \ldots, n-1).$$

5. The **trapezoidal method** for integrals, based on approximating the area under a segment of a curve by a **trapezium**, is

$$\int_a^b f(x)\, dx \simeq h(\tfrac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)),$$

where

$$h = (b - a)/n,$$
$$x_r = a + rh \quad (r = 0, 1, \ldots, n).$$

6. Simpson's method for integrals, based on approximating the area under a segment of a curve by a parabola, is

$$\int_a^b f(x)\, dx \simeq \tfrac{1}{3}h(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)),$$

where

$$h = (b - a)/n,$$
$$x_r = a + rh \quad (r = 0, 1, \ldots, n),$$

and n must be even.

7. The three integration formulas are examples of **Newton–Cotes** formulas, obtained by integrating an interpolating polynomial. They are said to be in **composite form** if $n > 1$ for Euler's and the trapezoidal formulas, and if $n > 2$ for Simpson's. For a given step size h, Euler's formula is normally the least accurate of the three and Simpson's the most accurate.

Unit 19 Numerical methods for differential equations

In all these methods the differential equation is taken to be

$$y' = m(x, y),$$

with $y(x_0)$ given, and we seek approximations to $y(x)$ for $x = x_1, x_2, \ldots$, where

$$x_r = x_0 + rh$$

and h is the step length or step size. The approximation to y_r, which means $y(x_r)$, is denoted by Y_r. The approximation to y'_r, which means $y'(x_r)$, is denoted by Y'_r and is given by the formula

$$Y'_r = m(x_r, Y_r).$$

Section 1: Numerical methods

1. By an **integration method** for the numerical solution of differential equations we mean a method based on a numerical integration formula.

2. Euler's method uses the recurrence relation

$$Y_{r+1} = Y_r + hY'_r.$$

3. The **trapezoidal method** uses the formula

$$Y_{r+1} = Y_r + \tfrac{1}{2}h(Y'_r + Y'_{r+1}).$$

In the case of a linear differential equation, for which

$$m(x, y) = l(x)y + k(x),$$

the formula of the trapezoidal method can be solved for Y_{r+1} to give

$$Y_{r+1} = \frac{(1 + \frac{1}{2}hl_r)Y_r + \frac{1}{2}h(k_r + k_{r+1})}{1 - \frac{1}{2}hl_{r+1}},$$

where $l_r = l(x_r)$, $k_r = k(x_r)$, and so on.

4. An **explicit** method is one where the recurrence relation for Y_{r+1} does not involve Y'_{r+1}, Y''_{r+1}, or higher derivatives at x_{r+1}. An **implicit** method is one where the recurrence relation does involve Y'_{r+1} or higher derivatives at x_{r+1}.

5. The **local truncation error** is the error introduced in a single step, estimated as the value of $Y_{r+1} - y_{r+1}$ when $Y_r = y_r$. The **principal term** in the local truncation error is the first non-vanishing term of the Taylor series about x_r in powers of h for the local truncation error; this series is calculated by subtracting the Taylor series

$$y_{r+1} = y_r + hy'_r + \tfrac{1}{2}h^2 y''_r + \cdots$$

from the corresponding series for Y_{r+1} obtained from the recurrence relation.

6. The principal term in the local truncation error for Euler's method is $-\frac{1}{2}h^2 y''_r$, for the Taylor series method of order n (with $n \geq 2$) it is $-h^{n+1}y_r^{(n+1)}/(n+1)!$, and for the trapezoidal method it is $\frac{1}{12}h^3 y'''_r$.

Section 2: The predictor-corrector method

1. The **Euler-trapezoidal predictor-corrector method** calculates Y_{r+1} in two stages:

$$Y^*_{r+1} = Y_r + hY'_r \qquad \text{where } Y'_r = m(x_r, Y_r),$$
$$Y_{r+1} = Y_r + \tfrac{1}{2}h(Y'_r + Y^{*\prime}_{r+1}) \qquad \text{where } Y^{*\prime}_{r+1} = m(x_{r+1}, Y^*_{r+1}).$$

The principal term in the local truncation error for this method involves h^3.

2. In general a **predictor-corrector method** is one where we calculate Y_{r+1}, the approximation to y_{r+1}, in two stages: the first stage uses an explicit method to give a first approximation Y^*_{r+1}, which is then used on the right-hand side in the recurrence relation of a more accurate implicit method to give an improved approximation Y_{r+1}.

Section 3: The analysis of numerical methods

1. A **one-step numerical method** for the solution of the differential equation $y' = m(x, y)$ is a method which approximates the differential equation by a first-order recurrence relation of the form

$$Y_{r+1} = Y_r + h\,\phi(x_r, Y_r, Y_{r+1}, h).$$

It is **consistent** with the differential equation if

$$\phi(x_r, y_r, y_r, 0) = m(x_r, y_r).$$

An alternative statement of the definition of consistency is

$$\lim_{h\to 0} \frac{Y_{r+1} - Y_r}{h} = m(x_r, Y_r).$$

2. For a fixed value of x, say x^*, the **global error** of a numerical method for calculating $y(x^*)$ using N steps of length $h = (x^* - x_0)/N$ is defined as $Y_N - y(x^*)$.

The method is said to be **convergent** on some interval $[x_0, b]$ if for all x^* in this interval (i.e. all x^* satisfying $x_0 \leq x^* \leq b$) we have

$$\lim_{h\to 0} Y_N = y(x^*).$$

3. A one-step method is convergent if and only if it is consistent. Further, if the principal term in the local truncation error is h^{p+1}, for some integer p, then the global error at x^* is given approximately by

$$Y_N - y(x^*) \simeq Ch^p,$$

where, for small h, C does not depend on h.

In particular, the global error for Euler's method is approximately proportional to h, whereas the global errors for the trapezoidal method, the Euler-trapezoidal method and the Taylor series method of order 2 are all approximately proportional to h^2 for small h.

4. A one-step method, applied to a given differential equation problem, is said to be **absolutely unstable** if the resulting recurrence relation problem is absolutely ill-conditioned, and to be **absolutely stable** if the recurrence relation problem is absolutely well-conditioned.

5. We can test the stability of a given numerical method by applying it to the **test problem** $y' = \alpha y$ with $y(x_0) = 1$, where α is a constant whose value is to be chosen. The **interval of absolute stability** of the method is the set of values of $h\alpha$ for which the method is absolutely stable.

6. The interval of absolute stability for Euler's method, for the Euler-trapezoidal predictor-corrector method and for the Taylor series method of order 2 is $(-2, 0)$, i.e. these methods are stable if $-2 < h\alpha < 0$. The interval of absolute stability for the trapezoidal method is $(-\infty, 0)$, i.e. the method is stable for all negative values of $h\alpha$.

7. Applied to the equation $y' = m(x, y)$, a given numerical method is absolutely stable if $h\dfrac{\partial m}{\partial y}(x_r, Y_r)$ lies in the interval of absolute stability for all values of r used in the calculation.

Section 4: Simpson's method

1. **Simpson's method** uses the recurrence relation

$$Y_{r+1} = Y_{r-1} + \tfrac{1}{3}h(Y'_{r-1} + 4Y'_r + Y'_{r+1}).$$

Since the recurrence relation is of second order, this may be called a **two-step method**.

2. Simpson's method is implicit, but if it is applied to a linear differential equation

$$y' = l(x)y + k(x),$$

the recurrence relation can be solved for Y_{r+1} to give

$$Y_{r+1} = \frac{4hl_rY_r + (3 + hl_{r-l})Y_{r-1} + h(k_{r-1} + 4k_r + k_{r+1})}{3 - hl_{r+1}},$$

where $l_r = l(x_r)$, $k_r = k(x_r)$, and so on.

3. The principal term in the local truncation error for Simpson's method contains the factor h^5, and the method is therefore capable of very high accuracy.

4. A **spurious solution** of the recurrence relation for Simpson's method is a term in the general solution which does not correspond to any solution of the differential equation. For the differential equation

$$y' = l(x)y + k(x),$$

the spurious solution is harmless provided $l(x)$ is positive, but if $l(x)$ is negative the spurious solution makes the method unstable.

Unit 20 Matrix algebra and determinants

Section 1: Some simple matrix operations

1. An $m \times n$ **matrix** is a rectangular array of numbers called **elements**, in m rows and n columns. The two numbers m and n together specify the **size** of the matrix. The element in the ith row and jth column of the matrix $\mathbf{A}$ can be denoted by a_{ij}. A **column matrix** or **column vector** is a matrix with just one column; a **row matrix** or **row vector** has just one row.

2. Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal** if they are the same size and corresponding elements in each matrix are equal.

3. If $\mathbf{A}$ is a matrix and k is a real number, then $k\mathbf{A}$ is the matrix of the same size whose elements are k times the corresponding elements of $\mathbf{A}$.

4. If two matrices $\mathbf{A}$ and $\mathbf{B}$ are the same size, then we can **add** them, the sum $\mathbf{A} + \mathbf{B}$ being the matrix formed by adding the corresponding elements of $\mathbf{A}$ and $\mathbf{B}$.

5. A matrix whose elements are all zero is called a **zero matrix** and is written $\mathbf{0}$. For any matrix $\mathbf{A}$ we have

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$$

and $$0\mathbf{A} = \mathbf{0}.$$

6. Given any matrix $\mathbf{A}$, the matrix $(-1)\mathbf{A}$, whose elements are the negatives of those of $\mathbf{A}$, is written $-\mathbf{A}$. This matrix has the property

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0},$$

and it can be used to define matrix subtraction as

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

7. The algebraic operation of adding matrices of a given size is closed, commutative and associative, and is distributive with respect to multiplication by a scalar (see Section 3.5 of this Handbook).

Multiplication by a scalar obeys the further rules

$$(k_1 k_2)\mathbf{A} = k_1(k_2\mathbf{A}),$$
$$(k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}.$$

Section 2: Matrix multiplication

1. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix, then we can multiply them, with $\mathbf{A}$ written on the left. The product $\mathbf{C} = \mathbf{AB}$ is an $m \times p$ matrix whose element in the ith row and jth column is

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

2. Matrix multiplication is associative, and is distributive with respect to matrix addition, i.e.

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C},$$
$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

and $$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

However, it is *not* (in general) commutative:

$$\mathbf{BA} \neq \mathbf{AB}.$$

3. The product of a matrix and a column vector, with the matrix on the left, is another column vector. A set of simultaneous linear algebraic equations can be written in the form

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A}$ is the (given) matrix of coefficients, $\mathbf{x}$ is a column vector whose elements are the unknowns, and $\mathbf{b}$ is a given column vector. Moreover, if $\mathbf{A}$ is a given matrix and $\mathbf{x}$ is a variable column vector of suitable size, then we can say that $\mathbf{A}$ **transforms** $\mathbf{x}$ to $\mathbf{Ax}$ (i.e. $\mathbf{A}$ defines a function whose domain is a set of column vectors; this function is called a **linear transformation**).

4. The **transpose** of a matrix $\mathbf{A}$, written $\mathbf{A}^T$, is obtained by interchanging the rows and columns of $\mathbf{A}$, i.e. the element in the ith row and jth column of $\mathbf{A}^T$ is a_{ji}. The main properties of transposed matrices are

$$(\mathbf{A}^T)^T = \mathbf{A},$$
$$(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{n-1}\mathbf{A}_n)^T = \mathbf{A}_n^T\mathbf{A}_{n-1}^T \ldots \mathbf{A}_2^T\mathbf{A}_1^T.$$

Section 3: Change of axes

Suppose Ox, Oy and $O'x', O'y'$ are two Cartesian coordinate systems in the plane, with origins O and O', that the origin O' of the second system has coordinates (a, b) with respect to the first system and that the direction of the $O'x'$ axis is obtained from that of Ox by an anticlockwise rotation through an angle α. Then the coordinates (x, y) of a point P with respect to the first system and its coordinates (x', y') with respect to the second are related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix}.$$

Section 4: Square matrices and their inverses

1. A **square matrix** is one with the same number of rows as columns. The **main diagonal** of a square matrix $\mathbf{A}$ consists of the elements $a_{11}, a_{22}, \ldots$.

2. A **unit** (or **identity**) **matrix**, written $\mathbf{I}$, is a square matrix with elements 1 on the main diagonal and 0 everywhere else. For any square matrix $\mathbf{A}$ and unit matrix $\mathbf{I}$ of the same size, we have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

3. A **singular** square matrix is one whose rows are linearly dependent. A **non-singular** square matrix is one whose rows are linearly independent. Every non-singular square matrix $\mathbf{A}$ has a unique **inverse** $\mathbf{A}^{-1}$ with the property

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

Singular matrices (and non-square matrices) do not have inverses.

4. To compute $\mathbf{A}^{-1}$, start with the matrix pair $\mathbf{A}|\mathbf{I}$ and perform the same Gaussian elimination type row operations on both matrices, including multiplying or dividing a row by a non-zero constant. The linear row operations which convert the left-hand matrix $\mathbf{A}$ to a unit matrix $\mathbf{I}$ will convert the right-hand matrix $\mathbf{I}$ to $\mathbf{A}^{-1}$.

In particular, the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has inverse

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

5. Matrix inverses have the property

$$(\mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_{n-1} \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} \mathbf{A}_{n-1}^{-1} \dots \mathbf{A}_2^{-1} \mathbf{A}_1^{-1}.$$

6. A square matrix $\mathbf{A}$ is said to be **symmetric** if $\mathbf{A}^T = \mathbf{A}$.

Section 5: Introduction to determinants

1. To every square matrix $\mathbf{A}$ there corresponds a number called its **determinant**, written $\det \mathbf{A}$ or $|\mathbf{A}|$.

2. If $\mathbf{A}$ is a 2×2 matrix, then

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

3. If $\mathbf{A}$ is a 3×3 matrix, then

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

An $n \times n$ determinant can be defined in terms of $(n-1) \times (n-1)$ determinants in a similar way.

4. The rows (or columns) of a square matrix $\mathbf{A}$ are linearly dependent if and only if $\det \mathbf{A} = 0$. Therefore $\mathbf{A}^{-1}$ exists if and only if $\det \mathbf{A} \neq 0$.

5. Linear combinations of rows (or columns) of a determinant can be added to any other row (or column) without changing the value of the determinant.

6. In an **upper triangular matrix**, all elements below the main diagonal are zero; in a **lower triangular matrix**, all elements above the main diagonal are zero; in a **diagonal matrix** all elements not on the main diagonal are zero. The determinant of an upper triangular, lower triangular or diagonal matrix is the product of the elements on its main diagonal.

7. To evaluate a determinant, use the operations of Gaussian elimination, namely adding or subtracting a multiple of one row from another and including row interchanges if necessary, to reduce the matrix to upper triangular form. Provided no row interchanges have been made, the upper triangular matrix has the same determinant as the original matrix; every interchange of a pair of rows reverses the sign of the determinant.

Unit 21 Eigenvalues and eigenvectors

Section 1: The theoretical eigenvalue problem

1. Given a square matrix $\mathbf{A}$ and a number λ, the equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

for the column vector $\mathbf{x}$ may have solutions other than the zero column vector. If it does, then we say that the number λ is an **eigenvalue** of the matrix $\mathbf{A}$, and that the non-zero column vector $\mathbf{x}$ is an **eigenvector** of $\mathbf{A}$. Any non-zero scalar multiple of an eigenvector is also an eigenvector, so there are an infinite number of eigenvectors of $\mathbf{A}$ corresponding to each eigenvalue.

2. One way of finding the eigenvalues λ of a given square matrix $\mathbf{A}$ is to solve the **characteristic equation**

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0,$$

which is a polynomial equation in λ. Eigenvectors can then be found by solving the equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

for $\mathbf{x}$.

3. If the characteristic equation has n different solutions $\lambda_1, \dots, \lambda_n$, where the size of the matrix $\mathbf{A}$ is $n \times n$, then the eigenvalues of $\mathbf{A}$ are said to be **distinct**. If not, then the polynomial $\det(\mathbf{A} - \lambda \mathbf{I})$ has at least one repeated linear factor $(\lambda - \lambda_i)^m$ with $m \geq 2$, and the corresponding eigenvalue λ_i is said to be **repeated**.

4. If the eigenvalues of an $n \times n$ matrix are distinct, then the matrix has n linearly independent eigenvectors. If the eigenvalues are not distinct, this may or may not be the case.

Section 2: Iterative methods for finding selected eigenvalues

1. If the square matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then for any real p, q:

(i) $\mathbf{A} + q\mathbf{I}$ has eigenvalues $\lambda_1 + q, \dots, \lambda_n + q$;

(ii) if $\mathbf{A}$ is non-singular, then $\mathbf{A}^{-1}$ has eigenvalues $\lambda_1^{-1}, \dots, \lambda_n^{-1}$;

(iii) if $\mathbf{A} - p\mathbf{I}$ is non-singular, then $(\mathbf{A} - p\mathbf{I})^{-1}$ has eigenvalues $(\lambda_1 - p)^{-1}, \dots, (\lambda_n - p)^{-1}$;

(iv) $\mathbf{A}$, $\mathbf{A} + q\mathbf{I}$, $\mathbf{A}^{-1}$ and $(\mathbf{A} - p\mathbf{I})^{-1}$, if they exist, all have the same eigenvectors.

2. Direct iteration. We use the recurrence relation

$$\mathbf{y}_{r+1} = \mathbf{A}\mathbf{y}_r / \alpha_{r+1},$$

where α_{r+1} is the element of largest modulus in $\mathbf{A}\mathbf{y}_r$. If $\mathbf{y}_0$ is not an eigenvector and not the zero vector, then as r becomes large, α_r approximates to the eigenvalue of $\mathbf{A}$ with largest modulus (assumed real and distinct) and $\mathbf{y}_r$ approximates to a corresponding eigenvector.

3. Inverse iteration. We calculate $\mathbf{y}'_{r+1}$ either from its definition $\mathbf{y}'_{r+1} = \mathbf{A}^{-1}\mathbf{y}_r$, or by solving the simultaneous equations

$$\mathbf{A}\mathbf{y}'_{r+1} = \mathbf{y}_r,$$

and then obtain $\mathbf{y}_{r+1}$ from

$$\mathbf{y}_{r+1} = \mathbf{y}'_{r+1}/\alpha_{r+1},$$

where α_{r+1} is the element of largest modulus in $\mathbf{y}'_{r+1}$.

For an arbitrary non-zero starting vector $\mathbf{y}_0$ which is not an eigenvector, and with $\mathbf{A}$ non-singular, the number $1/\alpha_r$ approximates to the eigenvalue with smallest modulus (assumed real and distinct) and $\mathbf{y}_r$ approximates to a corresponding eigenvector, for sufficiently large r.

4. Modified inverse iteration. To find the eigenvalue closest to p and a corresponding eigenvector of $\mathbf{A}$, we start with an arbitrary non-zero vector $\mathbf{y}_0$ which is not an eigenvector and calculate $\mathbf{y}'_{r+1}$ either from its definition $\mathbf{y}'_{r+1} = (\mathbf{A} - p\mathbf{I})^{-1}\mathbf{y}_r$ or by solving

$$(\mathbf{A} - p\mathbf{I})\mathbf{y}'_{r+1} = \mathbf{y}_r,$$

and then obtain $\mathbf{y}_{r+1}$ from

$$\mathbf{y}_{r+1} = \mathbf{y}'_{r+1}/\alpha_{r+1},$$

where α_{r+1} is the element of largest modulus in $\mathbf{y}'_{r+1}$.

If $\mathbf{A} - p\mathbf{I}$ is non-singular, and r is large enough, the number $p + (1/\alpha_r)$ approximates to the eigenvalue closest to p (assumed real and distinct), and $\mathbf{y}_r$ approximates to a corresponding eigenvector.

5. For a 3×3 matrix where we know the eigenvalues of largest and smallest modulus, denoted here by λ_1 and λ_2, a good choice of p in modified inverse iteration is either $\frac{1}{2}(|\lambda_1| + |\lambda_2|)$ or $-\frac{1}{2}(|\lambda_1| + |\lambda_2|)$.

Section 3: Decomposition methods for finding all the eigenvalues

1. To **decompose** a square matrix $\mathbf{A}$ is to express it as a product of two other square matrices, and the **LU decomposition** of $\mathbf{A}$ (when it exists) is

$$\mathbf{A} = \mathbf{LU},$$

where $\mathbf{L}$ is lower triangular with all diagonal elements equal to 1, and $\mathbf{U}$ is upper triangular.

In particular, a 2×2 matrix $\mathbf{A}$ has an $\mathbf{LU}$ decomposition

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & \det \mathbf{A}/a_{11} \end{bmatrix},$$

provided that a_{11} is not zero.

2. The eigenvalues of a diagonal, upper triangular or lower triangular matrix are equal to the elements on the main diagonal.

3. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{P}$ are $n \times n$ matrices, with $\mathbf{P}$ non-singular, then:

(i) $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues;

(ii) $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{AP}$ have the same eigenvalues. Further, if $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\mathbf{P}$ is the matrix whose columns are the corresponding eigenvectors of $\mathbf{A}$, then

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

where all off-diagonal elements of this last matrix are zero. Even if the eigenvalues are not all distinct, the result still holds provided a set of n linearly independent eigenvectors exists.

4. The **LR method** for calculating eigenvalues. Given a square matrix $\mathbf{A}$, form the sequence $\mathbf{A}_0, \mathbf{A}_1, \ldots$, as follows:

$$\mathbf{A}_0 = \mathbf{A},$$

$$\mathbf{A}_{r+1} = \mathbf{U}_r\mathbf{L}_r,$$

where $\mathbf{A}_r = \mathbf{L}_r\mathbf{U}_r$ is the $\mathbf{LU}$ decomposition of $\mathbf{A}_r$. For suitable $\mathbf{A}$, the sequence $\mathbf{A}_0, \mathbf{A}_1, \ldots$ will converge to an upper triangular matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$ arranged in order of decreasing modulus.

5. If the LR method applied to a matrix $\mathbf{A}$ fails to converge, try applying it to the matrix $\mathbf{A} + q\mathbf{I}$, where q is a real number. The eigenvalues of $\mathbf{A} + q\mathbf{I}$ are $\lambda_1 + q, \lambda_2 + q, \ldots$, where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of $\mathbf{A}$.

Unit 22 Simultaneous differential equations

Section 1: First-order systems

1. A **linear first-order system** of differential equations is one that can be written in the form

$$\mathbf{A}_1(t)\dot{\mathbf{x}}(t) + \mathbf{A}_2(t)\mathbf{x}(t) = \mathbf{g}(t),$$

where $\mathbf{x}(t)$ is a column vector whose elements are the unknown functions of t, $\dot{\mathbf{x}}(t)$ is the column vector obtained by differentiating these functions, $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ are given matrices and $\mathbf{g}(t)$ is a given column vector whose entries may depend on t.

A **linear second-order system** is one that can be written

$$\mathbf{A}_1(t)\ddot{\mathbf{x}}(t) + \mathbf{A}_2(t)\dot{\mathbf{x}}(t) + \mathbf{A}_3(t)\mathbf{x}(t) = \mathbf{g}(t).$$

If all the entries in the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, and $\mathbf{A}_3$ if present, are constants, we have a **constant-coefficient** system. If $\mathbf{g}(t) = \mathbf{0}$ for all t, we have a **homogeneous** system.

2. The **normal form** (if it exists) of a linear first-order constant-coefficient system is

$$\dot{\mathbf{x}}(t) = \mathbf{B}\mathbf{x}(t) + \mathbf{h}(t),$$

where $\mathbf{B}$ is a constant square matrix.

3. If $\mathbf{B}$ is an $n \times n$ matrix with n linearly independent eigenvectors, say $\mathbf{a}_1, \ldots, \mathbf{a}_n$, corresponding to (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution of the homogeneous system

$$\dot{\mathbf{x}}(t) = \mathbf{B}\mathbf{x}(t)$$

is

$$\mathbf{x}(t) = C_1\mathbf{a}_1e^{\lambda_1 t} + \cdots + C_n\mathbf{a}_ne^{\lambda_n t},$$

where $C_1, \ldots, C_n$ are arbitrary constants. If $\mathbf{B}$ is real then the expressions $\mathbf{a}_re^{\lambda_r t}$ in the general solution are either real or occur in complex conjugate pairs. If $\mathbf{a}e^{\lambda t}$ and $\overline{\mathbf{a}}e^{\overline{\lambda}t}$ is such a complex conjugate pair, then the real pair $\mathrm{Re}(\mathbf{a}e^{\lambda t})$ and $\mathrm{Im}(\mathbf{a}e^{\lambda t})$ may be used instead.

Section 2: Further methods for linear first-order systems

1. If $\mathbf{B}$ is an $n \times n$ matrix with n linearly independent eigenvectors, then the system

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{h}(t)$$

can be solved by putting

$$\mathbf{x} = \mathbf{P}\mathbf{y},$$

where $\mathbf{P}$ is a square matrix whose columns are the eigenvectors of $\mathbf{B}$. The resulting equation for $\mathbf{y}$ is

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\mathbf{y} + \mathbf{P}^{-1}\mathbf{h}(t)$$

and, since $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is a diagonal matrix (its diagonal elements being the eigenvalues of $\mathbf{B}$), we can solve the equations for the various elements of $\mathbf{y}$ individually. The solution $\mathbf{x}$ is then found from $\mathbf{x} = \mathbf{P}\mathbf{y}$.

2. Euler's method. This numerical procedure can be used for first-order systems of the form

$$\dot{\mathbf{x}} = \mathbf{m}(\mathbf{x}, t),$$

with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{m}$ is a column vector whose elements may depend on the elements of $\mathbf{x}$ and on t.

The approximation to $\mathbf{x}(t_r)$, where $t_r = t_0 + rh$ with h the step length, is denoted by

$$\mathbf{X}_r = [X_{1,r} \quad X_{2,r} \quad \dots \quad X_{n,r}]^T,$$

and we take $\mathbf{X}_0 = \mathbf{x}_0$. The recurrence relation is

$$X_{i,r+1} = X_{i,r} + hm_i(\mathbf{X}_r, t_r) \quad (i = 1, \dots, n),$$

or $\mathbf{X}_{r+1} = \mathbf{X}_r + h\mathbf{m}(\mathbf{X}_r, t_r)$.

Section 3: Second-order homogeneous systems

1. If $\mathbf{B}$ is an $n \times n$ real matrix with n linearly independent eigenvectors, the general solution of the system

$$\ddot{\mathbf{x}} = \mathbf{B}\mathbf{x}$$

is a sum of the following terms:

(i) for each real positive eigenvalue λ_r with eigenvector $\mathbf{a}_r$, a term

$$\mathbf{a}_r(C_r e^{\sqrt{\lambda_r}t} + D_r e^{-\sqrt{\lambda_r}t});$$

(ii) for a zero eigenvalue with eigenvector $\mathbf{a}_r$, a term

$$\mathbf{a}_r(C_r + D_r t);$$

(iii) for each real negative eigenvalue λ_r with eigenvector $\mathbf{a}_r$, a term

$$\mathbf{a}_r(C_r \cos\sqrt{-\lambda_r}t + D_r \sin\sqrt{-\lambda_r}t);$$

(iv) for each complex pair of eigenvalues λ_r and $\overline{\lambda_r}$, a term

$$C_r \operatorname{Re}(\mathbf{a}_r e^{\sqrt{\lambda_r}t}) + C'_r \operatorname{Im}(\mathbf{a}_r e^{\sqrt{\lambda_r}t})$$
$$+D_r \operatorname{Re}(\mathbf{a}_r e^{-\sqrt{\lambda_r}t}) + D'_r \operatorname{Im}(\mathbf{a}_r e^{-\sqrt{\lambda_r}t}),$$

where $\mathbf{a}_r$ is an eigenvector for λ_r.

Section 4: Forced oscillations

1. *Theorem 1:* Suppose that $\mathbf{x}_p$ is a particular solution of the system

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{h}(t). \qquad (1)$$

Then the general solution of this system is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_c,$$

where $\mathbf{x}_c$ (the complementary function) is the general solution of the associated homogeneous system

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{0}.$$

2. *Theorem 2:* Suppose the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are real, constant, symmetric and square, and that all their eigenvalues are positive. (A symmetric matrix is one that equals its own transpose.) Then every solution of

$$\mathbf{A}_1\ddot{\mathbf{x}}(t) + \mathbf{A}_2\dot{\mathbf{x}}(t) + \mathbf{A}_3\mathbf{x}(t) = \mathbf{0}$$

becomes small (i.e. every entry in the column vector $\mathbf{x}(t)$ becomes small) as t becomes large.

3. To look for a particular solution of

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{b}_1 \cos\omega t + \mathbf{b}_2 \sin\omega t,$$

write the system in the phasor form

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \operatorname{Re}(\mathbf{b}e^{i\omega t}),$$

where $\mathbf{b} = \mathbf{b}_1 - i\mathbf{b}_2$, and look for a solution of the form

$$\mathbf{x} = \operatorname{Re}(\mathbf{z}e^{i\omega t}),$$

where $\mathbf{z}$ is a column vector whose entries are complex constants.

Unit 24 Normal modes

Section 1: Modelling vibrating systems

1. A **lumped-parameter model** is a model of a real mechanical system in which each component of the model has only one of the three properties mass, stiffness and damping. The only components used in this unit are *particles* (having mass only) and *perfect springs* (having stiffness only).

2. The **number of degrees of freedom** in a lumped-parameter model is the number of coordinates necessary to specify the configuration (i.e. the positions of all the particles).

3. The behaviour of a vibrating system is said to be **forced** if time-dependent external forces act on it; otherwise the behaviour is said to be **free**.

Section 2: Free undamped vibrations with two degrees of freedom

1. The **equilibrium configuration** (or **static configuration**) of a lumped-parameter system is the configuration for which the total force on each particle is zero; an **equilibrium position** of a particle is its position when the system is in an equilibrium configuration.

2. A **normal mode** of a mechanical system is a motion of the system in which all the particles execute simple harmonic motion with the same angular frequency. The angular frequencies for which this is possible are called **normal mode angular frequencies**.

3. In normal mode motion, the displacements of any pair of particles from their equilibrium positions are in a constant ratio, called the **displacement ratio** for that pair of particles. If the displacement ratio is positive, the two particles are said to be **in phase**; if it is negative, they are said to be **phase opposed**.

4. The equations of motion for an undamped vibrating system can, by dividing each equation by the relevant mass, be put in the form

$$\ddot{\mathbf{x}} = \mathbf{H}\mathbf{x},$$

where $\mathbf{x}$ is a column vector whose entries are the displacements of the various particles from their equilibrium positions, and $\mathbf{H}$ is a square matrix with constant entries. Then each eigenvalue of $\mathbf{H}$ is $-\omega^2$, where ω is a normal mode angular frequency, and the corresponding eigenvector, if the top entry in it is chosen to be 1, gives the displacement ratios for the corresponding normal mode.

5. The most general motion of a vibrating system is an arbitrary linear combination of the normal mode motions.

Section 3: Vibration absorption

1. Forced vibrations. When a sinusoidally varying external force is applied to some part of a lumped-parameter system, a motion is in general possible for which all particles execute simple harmonic motion at the angular frequency of the external force. This **steady-state** motion can be used to model the steady-state response of a real system to such a force.

2. If the frequency of the external force is close to a normal mode frequency, then the forced vibrations exhibit **resonance**—the amplitude of the steady-state response is very large.

3. There may be a frequency for which one of the particles remains at rest in the steady state. This phenomenon can be used to isolate the object modelled by that particle from the effects of forced vibration at (or near) this frequency. The system is then said to act as a **vibration absorber**.

Section 4: Extending the scope

A **degenerate** system is one for which one of the normal mode angular frequencies is zero. In the corresponding motion each particle moves with the same constant velocity (rather than sinusoidally).

Unit 25 Functions of more than one variable

Section 1: First-order partial derivatives

1. Any expression or formula which involves two variables, say x and y, and whose value is uniquely determined by the values of x and y, is called a **function of the two variables** x and y. If a variable z is a function of x and y (i.e. $z = f(x, y)$, where f is a function of the two variables x and y) then we call x and y the **independent variables** and z the **dependent variable**. The set of possible pairs of values (x, y) for the two independent variables is called the **domain** of the function; it can be thought of as a region in the (x, y)-plane.

Functions of three (or more) variables are defined analogously.

2. A function of two variables, say $f(x, y)$, can be represented in three-dimensional space by the set of points whose coordinates x, y, z satisfy $z = f(x, y)$. Such a set of points is an example of a **surface**.

3. A **section function** is a function of one variable, obtained from a function of several variables by holding all but one of the independent variables fixed.

4. If $f(x, y)$ is a function of two variables, and a, b are two constants, then the **(first-order) partial derivative** of $f(x, y)$ with respect to x at (a, b) is defined as the (ordinary) derivative at a of the section function $f(x, b)$, and is denoted by $\dfrac{\partial f}{\partial x}(a, b)$. Similarly, the partial derivative of $f(x, y)$ with respect to y at (a, b) is the derivative at b (with respect to y) of the section function $f(a, y)$, and is denoted by $\dfrac{\partial f}{\partial y}(a, b)$.

5. We often use the partial derivatives of $f(x, y)$ at (x, y) instead of at (a, b). The partial derivative $\dfrac{\partial f}{\partial x}(x, y)$ is calculated by differentiating $f(x, y)$ with respect to x while treating y as a constant. This procedure is called **differentiating** $f(x, y)$ **partially** with respect to x. Similarly, $\dfrac{\partial f}{\partial y}$ is calculated by differentiating $f(x, y)$ partially with respect to y at constant x.

First-order partial derivatives of functions of three or more variables are defined similarly.

Section 2: Some applications of partial derivatives

1. Let $f(x, y)$ be a **well-behaved** function—that is, one such that the surface $z = f(x, y)$ is smooth in the sense of having no folds or breaks. Then the **Taylor polynomial of first order** for f at (a, b) is

$$p_1(x, y) = f(a, b) + (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b).$$

2. The first-order **Taylor approximation** formula at (a, b), $f(x, y) \simeq p_1(x, y)$, holds for (x, y) close to (a, b). It can be written

$$f(x, y) \simeq f(a, b) + (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b)$$

or $$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y,$$

where

$$\delta f = f(x, y) - f(a, b),$$
$$\delta x = x - a,$$
$$\delta y = y - b.$$

3. The **triangle inequality** is

$$|a + b| \leq |a| + |b|$$

and holds for any two numbers a and b.

4. The **tangent plane** to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ has the equation $z = p_1(x, y)$.

5. The **chain rule** for the function of two variables $f(x, y)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

where x and y are functions of the single variable t.

6. The above formulas can be generalized to functions of three or more variables.

Section 3: Higher-order partial derivatives and Taylor polynomials

1. The **second-order partial derivatives** of a function $f(x, y)$ are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

Third- and higher-order partial derivatives are defined similarly, as are the partial derivatives of functions of more than two variables.

2. The **Mixed Derivative Theorem**. If $f(x, y)$ is a well-behaved function of two variables, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

3. The **Taylor polynomial of second order** for $f(x, y)$ at (a, b) is

$$p_2(x, y) = f(a, b) + (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b) + \tfrac{1}{2}(x - a)^2 \frac{\partial^2 f}{\partial x^2}(a, b) + (x - a)(y - b)\frac{\partial^2 f}{\partial x \partial y}(a, b) + \tfrac{1}{2}(y - b)^2 \frac{\partial^2 f}{\partial y^2}(a, b).$$

4. For (x, y) near to (a, b), the **second-order Taylor approximation** at (a, b) is $f(x, y) \simeq p_2(x, y)$ and is normally more accurate than the first-order Taylor approximation.

Section 4: Maxima and minima

1. If, for all x and y sufficiently close to a and b,

$$f(x, y) \leq f(a, b),$$

then f has a (**local**) **maximum** at the point (a, b); but if the inequality is

$$f(x, y) \geq f(a, b),$$

then f has a (**local**) **minimum** at (a, b).

2. An **extremum** is a point which is either a local maximum or a local minimum.

3. A **stationary point** of a function $f(x, y)$ is a point (a, b) where

$$\frac{\partial f}{\partial x}(a, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = 0.$$

The **stationary point criterion**. If (a, b) is an extremum of a well-behaved function $f(x, y)$, and is not on the boundary of the domain of $f(x, y)$, then (a, b) is a stationary point of $f(x, y)$. (The converse statement is not necessarily true.)

4. A **saddle point** is a stationary point which is not an extremum.

5. The $AC - B^2$ **criterion**. Suppose (a, b) is a stationary point of $f(x, y)$, and let

$$A = \frac{\partial^2 f}{\partial x^2}(a, b), \quad B = \frac{\partial^2 f}{\partial x \partial y}(a, b), \quad C = \frac{\partial^2 f}{\partial y^2}(a, b).$$

Then

(i) if $AC - B^2 > 0$, (a, b) is a $\begin{cases} \text{maximum} & \text{if } A < 0, \\ \text{minimum} & \text{if } A > 0; \end{cases}$

(ii) if $AC - B^2 < 0$, (a, b) is a saddle point;

(iii) if $AC - B^2 = 0$, the criterion gives no information.

Unit 26 Vector calculus

Section 1: Scalar and vector fields

1. A **scalar field function**, or **scalar field**, is a function which to each point in a given region of space associates a unique scalar. The region is the **domain** of the scalar field.

2. The **contour curves** (or **contours**) of a two-dimensional scalar field $\phi(x, y)$ are the curves $\phi(x, y) = \text{constant}$. The **contour surfaces** of a three-dimensional scalar field $\phi(x, y, z)$ are the surfaces $\phi(x, y, z) = \text{constant}$.

3. A **vector field function**, or **vector field**, is a function which to every point in a given region of space (or of the plane) associates a unique vector.

4. The **field lines** of a vector field are the family of continuous curves such that at every point the tangent to the curve is along the direction of the vector field at that point.

Section 2: Differentiating scalar fields

1. The symbol ∇, defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

is a **differential vector operator**.

2. For a three-dimensional scalar field $\phi(x, y, z)$, the vector field

$$\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

is called the **gradient** of ϕ and is denoted by $\operatorname{grad}\phi$ or $\nabla\phi$.

3. The direction of $\operatorname{grad}\phi$ at a point P in the domain of the scalar field ϕ is perpendicular to the contour curve or surface passing through P. This direction is the **normal** to this contour at P.

4. The vector field $\operatorname{grad}\phi$ gives the magnitude and direction of the maximum (spatial) rate of change of the scalar field ϕ at any point P in the domain of the scalar field.

5. The (spatial) rate of change of a scalar field ϕ at a point P in the direction of a unit vector $\mathbf{e}$ is given by $\mathbf{e} \cdot (\operatorname{grad}\phi)$.

Section 3: The scalar line integral

1. The **scalar line integral** of a vector field $\mathbf{F}$ along a curve C, called the **path of integration**, with end-points A and B, is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \lim \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta\mathbf{r}_i,$$

where the curve is divided into n segments with end-points $P_1(=A), P_2, P_3, \ldots, P_{n+1}(=B)$; $\mathbf{F}_i$ means $\mathbf{F}(P_i)$; $\delta\mathbf{r}_i$ means $\overrightarrow{P_iP_{i+1}}$; and the limit is taken in such a way that n becomes very large and the segments all become very short. The end-points A and B may coincide, in which case the path of integration is **closed**; if they do not coincide, it is **open**.

2. A three-dimensional curve can be expressed in terms of a parameter t by the **parametric equations**

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \leq t \leq b).$$

3. To evaluate a scalar line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$:

(i) describe the path C by means of a set of parametric equations;

(ii) use these parametric equations and the formula

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

to express the vector field $\mathbf{F}$ and $d\mathbf{r}/dt$ in terms of the parameter t, and hence rewrite the line integral as an integral with respect to t:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt}\, dt;$$

(iii) evaluate this integral with respect to t.

4. The line integral of $\operatorname{grad}\phi$ is independent of the path apart from its end-points:

$$\int_{AB} (\operatorname{grad}\phi) \cdot d\mathbf{r} = \phi(B) - \phi(A).$$

5. A vector field $\mathbf{F}$ is **conservative** if for every closed path C in the domain of $\mathbf{F}$ we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$$

where the symbol $\displaystyle\oint_C$ means a line integral round the closed path C. Otherwise the vector field is called **non-conservative**.

Section 4: The curl of a vector field

1. For a vector field $\mathbf{F}$ we define the **component of curl $\mathbf{F}$** at a point P in the direction of a unit vector $\mathbf{n}$ to be

$$\mathbf{n} \cdot (\operatorname{curl}\mathbf{F}) = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is a small closed curve in the plane through P perpendicular to $\mathbf{n}$, which contains P and encloses an area A. (The direction of integration round C is chosen so that a screw turned in this direction advances along $\mathbf{n}$.)

2. The curl of a vector field $\mathbf{F}(x, y, z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ can be evaluated by using the derivative formula

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

3. The curl of a vector field $\mathbf{F}$ is a detector for rotational motion; for instance, the curl of a fluid velocity field describes the spin of a small float placed in the fluid.

4. If $\mathbf{F}$ is a conservative vector field:

(i) $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in the domain of $\mathbf{F}$;

(ii) the line integral $\displaystyle\int_{AB} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path taken between any two points A and B;

(iii) $\operatorname{curl}\mathbf{F} = \mathbf{0}$;

(iv) there exists a scalar field ϕ such that $\mathbf{F} = \operatorname{grad}\phi$;

(v) $\displaystyle\int_{AB} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)$, where $\mathbf{F} = \operatorname{grad}\phi$.

5. Provided that the domain of $\mathbf{F}$ is sufficiently simple in shape, the converse of property 4(iii) above is also true, i.e. the equation $\operatorname{curl}\mathbf{F} = \mathbf{0}$ implies that $\mathbf{F}$ is conservative.

Unit 27 Multiple integrals

Section 1: The surface integral

1. Suppose that f is a two-dimensional scalar field whose domain includes a region S of the (x, y)-plane. We divide the region S into a number N of area elements so that the ith element has area δA_i, and denote some point in the ith element by P_i. Then the **surface integral** of f over the region S is defined to be

$$\int_S f\, dA = \lim \sum_{i=1}^{N} f(P_i)\delta A_i,$$

where the limit is taken in such a way that N becomes very large and the sizes of all the elements become very small. We call S the **region of integration**.

2. The following procedure can be used to evaluate surface integrals.

(i) Draw a diagram showing the region of integration, including the equations of its boundary lines.

(ii) Mark on the diagram the minimum and maximum values of x, which we call here a and b respectively.

(iii) Draw a typical strip of the region parallel to the y-axis and denote the formulas for the minimum and maximum values of y (which depend on the x-coordinate of the strip) by $\alpha(x)$ and $\beta(x)$ respectively.

(iv) The surface integral can now be written in terms of two single integrals:

$$\int_S f\, dA = \int_{x=a}^{x=b} \left[\int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\, dy\right] dx.$$

(v) Evaluate the 'inner' integral, holding x constant:

$$g(x) = \int_{y=\alpha(x)}^{y=\beta(x)} f(x,y)\,dy.$$

(vi) Evaluate the 'outer' integral:

$$\int_S f\,dA = \int_{x=a}^{x=b} g(x)\,dx.$$

An analogous procedure, using strips parallel to the x-axis, can also be used in which we first integrate over x and then integrate over y.

Section 2: Multiple integrals useful in mechanics

1. Suppose that f is a three-dimensional scalar field whose domain includes a region B of space. The **volume integral** of f over the region B is defined as

$$\int_B f\,dV = \lim \sum_{i=1}^{N} f(P_i)\delta V_i,$$

where B is divided into a number N of volume elements, the ith of which has volume δV_i and contains the point P_i, and the limit is taken in such a way that all the volume elements become very small. We call B the **region of integration**.

2. If ρ is the scalar field giving the local density of a body occupying a region B, then the mass of the body is

$$M = \int_B \rho\,dV.$$

The moment of inertia of this same body about some given axis is

$$I = \int_B (\rho d^2)\,dV,$$

where d denotes distance from this axis.

3. The mass of a hollow sphere of uniform density ρ with external radius a and internal radius b is $\frac{4}{3}\pi\rho(a^3 - b^3)$, and its moment of inertia about an axis through its centre is $\frac{8}{15}\pi\rho(a^5 - b^5)$. The corresponding formulas for a solid sphere are obtained by setting $b = 0$.

4. To evaluate a surface integral $\int_S f\,dA$ using polar coordinates (r,θ) the following procedure can be used.

(i) Draw a diagram showing the region of integration, S, and its bounding lines.

(ii) Write the surface integral in terms of r and θ using the formulas

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$\delta A = r\,\delta r\,\delta\theta.$$

In the limit of very small area elements, this last formula becomes $dA = r\,dr\,d\theta$.

(iii) Find the minimum and maximum values of θ in S, say θ_1 and θ_2.

(iv) For a strip of fixed θ, indicate the minimum and maximum values of r, say a and b (here assumed independent of θ, for simplicity).

(v) Write the surface integral in terms of two single integrals:

$$\int_{\theta=\theta_1}^{\theta=\theta_2} \left[\int_{r=a}^{r=b} f(r,\theta)\,r\,dr\right] d\theta.$$

(vi) Evaluate the inner integral, then the outer integral.

Section 3: The volume integral

The following procedure can be used to evaluate a volume integral $\int_B f\,dV$ using Cartesian coordinates.

(i) Draw two diagrams, showing (a) the region B of integration and (b) the projection S of this region on the (x,y)-plane. (The projection of a point (x,y,z) on the (x,y)-plane is the point (x,y), and the projection of B comprises the projections of all the points in B.)

(ii) Within the region B, draw a column parallel to the z-axis and mark its end-points, which are the equations of the top and bottom surfaces of B written in the form $z = \psi(x,y)$ and $z = \phi(x,y)$ respectively.

(iii) Evaluate the single integral

$$g(x,y) = \int_{z=\phi(x,y)}^{z=\psi(x,y)} f(x,y,z)\,dz.$$

(iv) Evaluate the surface integral of $g(x,y)$ over the plane region S. The result is $\int_B f\,dV$.

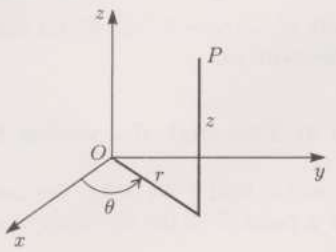
Similar procedures, with different orders of integration, can also be used.

Section 4: Changing variables in three dimensions

1. The **cylindrical polar coordinates** (r,θ,z) of a point P in space are related to its Cartesian coordinates (x,y,z) by

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z,$$

where $r \geq 0$, $0 \leq \theta < 2\pi$, $-\infty < z < \infty$. That is, (r,θ) are the plane polar coordinates of the projection of P onto the (x,y)-plane.



The volume element in cylindrical polar coordinates is

$$\delta V = r\,\delta r\,\delta\theta\,\delta z.$$

This becomes $dV = r\,dr\,d\theta\,dz$ in the limit.

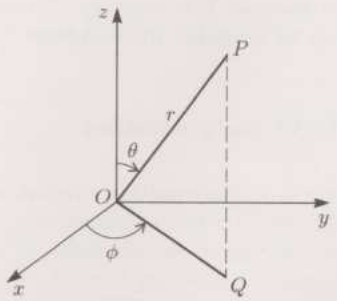
2. To evaluate a volume integral using cylindrical polar coordinates, a procedure similar to the one given above for Cartesian coordinates can be used; the only difference

is that we use plane polar coordinates r, θ in place of plane Cartesian coordinates x, y throughout.

3. The **spherical polar coordinates** (r, θ, ϕ) of a point P are related to its Cartesian coordinates (x, y, z) by

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta,$$

where $r \geq 0$, $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$. In this case r is the distance from P to the origin O, θ is the angle between the line OP and the positive z-axis, and ϕ is the angle between OQ and the positive x-axis, where Q is the projection of P onto the (x, y)-plane.



The volume element in spherical polar coordinates is

$$\delta V = r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi.$$

This becomes $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$ in the limit.

4. To evaluate a volume integral using spherical polar coordinates, use a procedure similar to the one given for Cartesian coordinates by expressing the volume integral as three repeated single integrals.

Unit 28 Moments and circular motion

Section 1: The components of a force

1. The component of a force $\mathbf{F}$ in a direction making an angle α with that of the force is equal to $F\cos\alpha$, where $F = |\mathbf{F}|$.

2. If a particle is static (in equilibrium), then the sum of the components of the forces acting on the particle in any direction is equal to zero.

Section 2: The moment of a force

1. The magnitude Γ of the **moment** of a force about a fixed point O is the product of the magnitude F of the force and the perpendicular distance d from the point O to the line of action of the force, that is,

$$\Gamma = Fd.$$

2. The **total moment** of the forces acting in two dimensions on a body about any point is the sum of the anticlockwise moments minus the sum of the clockwise moments.

3. If a rigid body is static (in equilibrium), then both the total force on the body and the total moment of forces acting on the body about any point are zero.

Section 3: Motion in a circle

1. In terms of polar coordinates, the *radial* and *transverse* components of the position, velocity and acceleration of a particle moving in a circle, r = constant, are as follows.

	Radial component	Transverse component
Position	r	0
Velocity	0	$r\dot{\theta}$
Acceleration	$-r\dot{\theta}^2$	$r\ddot{\theta}$

2. For circular motion with radius r and **angular velocity** $\omega = \dot{\theta}$, the velocity of the particle is $v = r\omega$, tangential to the circle. Both v and ω are positive for anticlockwise motion. The component of acceleration towards the centre of the circle is $r\omega^2 = v^2/r$. For non-uniform motion, the acceleration also has a transverse component $r\dot{\omega}$.

3. For *uniform* circular motion, with radius r and constant angular velocity ω, the acceleration has magnitude $r\omega^2 = v^2/r$ and is directed towards the centre of the circle. The time taken for one complete revolution of the circle is $\tau = 2\pi/|\omega|$.

Section 4: Using vector notation

1. The **torque** $\boldsymbol{\Gamma}$, about a fixed point O, of a force $\mathbf{F}$ acting at a point which has position vector $\mathbf{r}$ relative to the point O, is defined to be

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}.$$

2. The torque of all the gravitational forces acting on a system of particles or extended body can be evaluated by considering the total weight of the system to be concentrated at its centre of mass.

3. If a system of particles or extended body is in equilibrium, then the total external force is equal to zero, as is the total torque of the external forces about any fixed point.

4. With respect to plane polar coordinates $[r, \theta]$, the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (in the radial and transverse directions, respectively) are given by

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

The vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}$ form a right-handed triad of unit vectors. The directions of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ vary with time, so that

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r.$$

5. The position, velocity and acceleration of a particle moving in a circle of constant radius r in the (x, y)-plane, with centre at the origin, are

$$\mathbf{r} = r\mathbf{e}_r,$$
$$\mathbf{v} = \dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta$$
and $$\mathbf{a} = \ddot{\mathbf{r}} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta.$$

6. The **angular velocity** $\boldsymbol{\omega}$ for a particle in circular motion is a vector whose magnitude is equal to the angular speed, and whose direction is along the axis of rotation, in the sense in which the rotation would drive a right-handed screw. If the unit vector $\mathbf{k}$ is perpendicular to the plane of motion and the centre of the circle lies on the z-axis, then

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k}.$$

7. For circular motion with angular velocity $\boldsymbol{\omega}$, the velocity of a particle is given by

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r},$$

where $\mathbf{r}$ is the position vector of the particle relative to an origin on the axis of rotation.

Unit 29 Angular momentum and rigid bodies

Section 1: Angular momentum

1. The **strong form of Newton's third law** states that for a pair of interacting particles, the force $\mathbf{F}_{12}$ on Particle 1 due to Particle 2 is equal in magnitude but opposite in direction to the force $\mathbf{F}_{21}$ on Particle 2 due to Particle 1, that is,

$$\mathbf{F}_{12} = -\mathbf{F}_{21},$$

and each of these forces acts along the line which joins the two particles.

2. For a particle which has position vector $\mathbf{r}$ relative to an origin O and linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$, the **angular momentum** $\boldsymbol{l}$ relative to O is

$$\boldsymbol{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}.$$

3. The **torque law** for a single particle states that the rate of change of the particle's angular momentum relative to a fixed point is equal to the applied torque relative to that point, that is,

$$\dot{\boldsymbol{l}} = \boldsymbol{\gamma}.$$

In the special case where the total torque acting on the particle is zero, the angular momentum of the particle about the fixed point is constant. In other words, if $\boldsymbol{\gamma} = \mathbf{0}$ then $\boldsymbol{l}$ is constant. This is the **law of conservation of angular momentum** for a single particle.

4. The **total angular momentum** $\mathbf{L}$ of a system of n particles, relative to a given origin, is the vector sum of the angular momenta of the individual particles, that is,

$$\mathbf{L} = \sum_{i=1}^{n} \boldsymbol{l}_i = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i),$$

where $\mathbf{r}_i$ is the position vector of the ith particle and m_i is its mass.

5. The **total external torque** $\boldsymbol{\Gamma}^{\text{ext}}$ acting on a system of n particles, relative to a given origin, is defined to be the vector sum of the external torques which act upon the individual particles, that is,

$$\boldsymbol{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \boldsymbol{\gamma}_i^{\text{ext}}.$$

6. The **torque law** for a system of particles states that the rate of change of the total angular momentum about the origin is equal to the total external torque about the origin, that is,

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}.$$

This law applies to any system for which the inter-particle forces satisfy the strong form of Newton's third law.

In particular, if the total external torque is zero then the total angular momentum $\mathbf{L}$ is constant. This is the **law of conservation of angular momentum** for a system of particles.

Section 2: Rigid-body rotation

1. A **rigid body** is a many-particle system with the property that all the inter-particle distances remain constant in time. The inter-particle forces in a rigid body are assumed to satisfy the strong form of Newton's third law, so that the torque law $\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$ applies.

2. **Rigid-body rotation** about a fixed axis is a rigid-body motion for which particles on the axis remain fixed, while other particles move in circles centred on and perpendicular to the axis. If the axis of rotation is chosen to be the z-axis, then all particles have the same angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. If $\mathbf{r}_i$ is the position of Particle i, then its velocity is given by

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i = \dot{\theta}\mathbf{k} \times \mathbf{r}_i.$$

3. The z-component of the total angular momentum $\mathbf{L}$ of the body described above is

$$L_z = I\dot{\theta},$$

where

$$I = \sum_{i=1}^{n} m_i d_i^2$$

is the **moment of inertia** of the body about the axis of rotation (the sum being taken over all particles of the body). It follows from the torque law that

$$I\ddot{\theta} = \dot{L}_z = \Gamma_z^{\text{ext}}.$$

The kinetic energy of the body is given by

$$T = \tfrac{1}{2} I\dot{\theta}^2.$$

4. The moment of inertia of a continuous rigid body about a given axis of rotation is

$$I = \int_B \rho(\mathbf{r}) d^2(\mathbf{r})\, dV,$$

where $\rho(\mathbf{r})$ is the density of mass at point $\mathbf{r}$, $d(\mathbf{r})$ is the distance of the point $\mathbf{r}$ from the axis of rotation, and the integration is over the volume of the body B.

For a homogeneous body, the density ρ is constant. The moments of inertia of certain homogeneous rigid bodies about axes through their centres of mass are given in Table 1.

Table 1: Moments of inertia of homogeneous rigid bodies

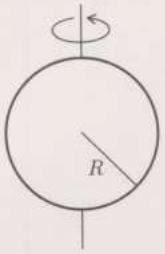
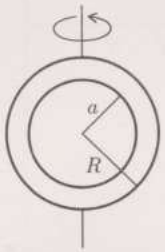
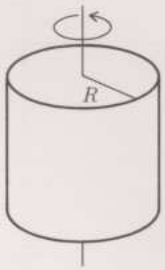
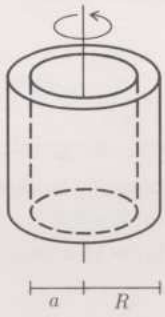
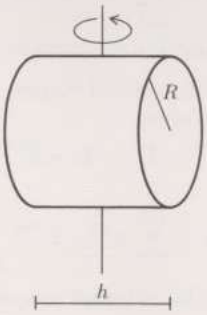
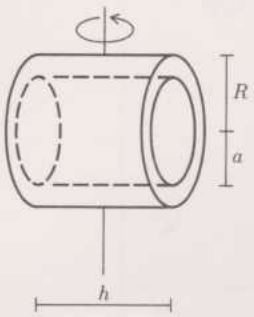
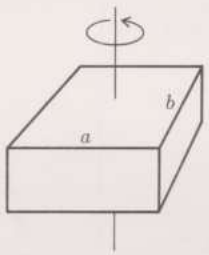
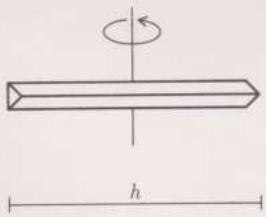
Homogeneous rigid body of mass M; axis	Diagram	Dimensions	Moment of inertia about given axis
solid sphere; axis through centre of mass		radius R	$\frac{2}{5}MR^2$
hollow sphere; axis through centre of mass		inner radius a, outer radius R	$\frac{2}{5}M\left(\frac{R^5-a^5}{R^3-a^3}\right)$ $\simeq \frac{2}{3}MR^2$ if $R \simeq a$
solid cylinder; axis through centre of mass and along axis of cylinder		radius R	$\frac{1}{2}MR^2$
hollow cylinder; axis through centre of mass and along axis of cylinder		inner radius a, outer radius R	$\frac{1}{2}M(R^2+a^2)$ $\simeq MR^2$ if $R \simeq a$
solid cylinder; axis through centre of mass and perpendicular to axis of cylinder		radius R, length h	$\frac{1}{4}MR^2 + \frac{1}{12}Mh^2$

Table 1 (continued)

Homogeneous rigid body of mass M; axis	Diagram	Dimensions	Moment of inertia about given axis
hollow cylinder; axis through centre of mass and perpendicular to axis of cylinder		inner radius a, outer radius R, length h	$\frac{1}{4}M(R^2+a^2)+\frac{1}{12}Mh^2$ $\simeq \frac{1}{2}MR^2+\frac{1}{12}Mh^2$ if $R \simeq a$
solid rectangular brick; axis through centre of mass and perpendicular to one pair of faces		faces perpendicular to axis have sides of lengths a and b	$\frac{1}{12}M(a^2+b^2)$
thin rod (cross-section of arbitrary shape); axis through centre of mass and perpendicular to rod		length h	$\frac{1}{12}Mh^2$

5. The **parallel axes theorem** states that the moment of inertia of a rigid body about an axis A can be written as

$$I = I_G + Md^2,$$

where M is the mass of the body, d is the distance of the centre of mass G from the axis A, and I_G is the moment of inertia of the body about an axis B which passes through G and is parallel to A.

Section 3: Rotation about a moving axis with fixed orientation

1. Consider a system of n particles, with centre of mass $\mathbf{R}$, total mass M, total kinetic energy T and total angular momentum $\mathbf{L}$ about the origin, which is acted upon by a total external force $\mathbf{F}^{\text{ext}}$ corresponding to a total external torque $\mathbf{\Gamma}^{\text{ext}}$ about the origin. Suppose that the ith particle has mass m_i and position $\mathbf{r}_i$, and experiences an external force $\mathbf{F}_i^{\text{ext}}$. Then the position of the particle relative to the centre of mass, $\mathbf{r}_i^{\text{rel}}$, is defined by

$$\mathbf{r}_i^{\text{rel}} = \mathbf{r}_i - \mathbf{R}.$$

The **centre of mass decomposition theorems** are as follows.

(i) The total kinetic energy of the system is given by

$$T = T_G + \tfrac{1}{2}M|\dot{\mathbf{R}}|^2,$$

where

$$T_G = \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{r}}_i^{\text{rel}}|^2$$

is the kinetic energy relative to the centre of mass.

(ii) The total angular momentum of the system about the origin is given by

$$\mathbf{L} = \mathbf{L}_G + \mathbf{R} \times M\dot{\mathbf{R}},$$

where

$$\mathbf{L}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times m_i\dot{\mathbf{r}}_i^{\text{rel}}$$

is the angular momentum about the centre of mass.

(iii) The total external torque of the system about the origin is given by

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{\Gamma}_G + \mathbf{R} \times \mathbf{F}^{\text{ext}},$$

where

$$\mathbf{\Gamma}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i^{\text{ext}}$$

is the total external torque about the centre of mass.

2. For a rigid body rotating with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$ about an axis which passes through the centre of mass and is always aligned with the z-axis, the kinetic energy

and the z-component of the total angular momentum relative to the centre of mass are given respectively by

$$T_G = \tfrac{1}{2}I\dot{\theta}^2 \quad \text{and} \quad \mathbf{L}_G \cdot \mathbf{k} = I\dot{\theta},$$

where I is the moment of inertia of the body about the axis of rotation.

3. For any system of particles, the rate of change of the total angular momentum relative to the centre of mass is equal to the total external torque relative to the centre of mass, that is,

$$\dot{\mathbf{L}}_G = \boldsymbol{\Gamma}_G.$$

This is the **torque law relative to the centre of mass**.

4. If a cylinder or sphere of radius R rolls (without slipping) along a rough plane, then the speed v of the centre is related to the angular speed $|\dot{\theta}|$ of the body about its centre by the **rolling condition**

$$v = R|\dot{\theta}|.$$

Section 4: Rotation about an axis whose orientation changes

1. If a body spins around an axis of symmetry (the *spin axis*), and this axis is fixed, then the total angular momentum of the body is given by

$$\mathbf{L} = I\dot{\theta}\,\mathbf{n},$$

where I is the moment of inertia with respect to the spin axis and $\dot{\theta}\mathbf{n}$ is the angular velocity of the body about this axis.

2. If a body spins around an axis of symmetry which is not fixed, and $I, \dot{\theta}\mathbf{n}$ are defined with respect to the spin axis as above, then the approximation

$$\mathbf{L} \simeq I\dot{\theta}\,\mathbf{n}$$

is valid provided that $|\dot{\mathbf{n}}|$ is small compared with $\dot{\theta}$.

3. The angular momentum of a spinning top has a constant magnitude and a constant vertical component. It satisfies the equation

$$\dot{\mathbf{L}} = \dot{\alpha}\,\mathbf{k} \times \mathbf{L},$$

where $\dot{\alpha}$ is the *rate of precession* of the spin axis about the vertical axis, and $\mathbf{k}$ is a unit vector in the vertically upward direction. If the top has mass M, moment of inertia I and angular velocity $\dot{\theta}\mathbf{n}$ about the spin axis, then the rate of precession is given by

$$\dot{\alpha} \simeq \frac{Mgc}{I\dot{\theta}},$$

where c is the distance of the centre of mass from the point of contact with the ground.

Unit 30 Planetary orbits

Section 1: Review of energy and planar motion

1. For particle motion in three or fewer dimensions, a force $\mathbf{F}$ acting on the particle is **conservative** if there exists a scalar field function $U = U(\mathbf{r}) = U(x, y, z)$ such that

$$\mathbf{F} = -\operatorname{grad} U.$$

The function U is the **potential energy** of the particle. In such a case, the total mechanical energy of the particle,

$$E = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

(where $\dot{\mathbf{r}}^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = |\dot{\mathbf{r}}|^2$), is constant throughout the motion.

2. In one dimension, the condition for a conservative force and the expression for the total mechanical energy become respectively

$$F = -\frac{dU}{dx} \quad \text{and} \quad E = \tfrac{1}{2}m\dot{x}^2 + U,$$

where F is the x-component of the force and $U = U(x)$.

3. The unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ corresponding to the plane polar coordinates $[r, \theta]$ (where $x = r\cos\theta$, $y = r\sin\theta$) are related to the Cartesian unit vectors $\mathbf{i}, \mathbf{j}$ by the equations

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

and

$$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta, \quad \mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta.$$

4. The derivatives $\dot{\mathbf{e}}_r, \dot{\mathbf{e}}_\theta$ are given in terms of $\mathbf{e}_r, \mathbf{e}_\theta$ by

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r.$$

5. The position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of a particle moving in the (x, y)-plane are given in terms of plane polar coordinates by the equations

$$\mathbf{r} = r\mathbf{e}_r,$$

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{e}_\theta.$$

Section 2: Kepler's laws and Newton's law of gravitation

1. Kepler's laws of planetary motion are as follows.

(i) Each planet moves in an ellipse, with the Sun at one focus.

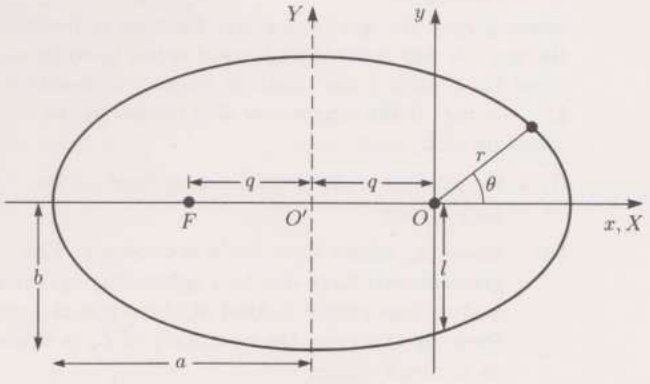
(ii) The line joining a planet to the Sun sweeps out equal areas in equal times.

(iii) The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

2. (i) The **ellipse** shown in the figure has *semi-major axis* a and *semi-minor axis* b. With respect to the axes $O'XY$, its equation is

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,$$

where $a \geq b > 0$.



(ii) The area of the ellipse is πab.

(iii) The *foci* of the ellipse are located a distance q to either side of the centre O', where

$$q = \sqrt{a^2 - b^2}.$$

(iv) With respect to axes Oxy with origin at the right-hand focus, and in polar coordinates, the equation of the ellipse is

$$\frac{l}{r} = 1 + e\cos\theta,$$

where

$$l = \frac{b^2}{a}$$

is the *semi-latus rectum*, and

$$e = \frac{q}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$

is the *eccentricity*, where $0 \leq e < 1$.

3. Newton's universal law of gravitation states that the gravitational force on a particle with mass m_i and position vector $\mathbf{r}_i$ due to another particle with mass m_j and position vector $\mathbf{r}_j$ is

$$\mathbf{F}_{ij} = \frac{Gm_im_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i),$$

where G is the *gravitational constant.*

4. A body has a **spherically symmetric** mass distribution if (with the origin of coordinates at the centre of the body) its mass density function $\rho(\mathbf{r})$ depends only on the radial distance $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$. The gravitational force exerted by such a body B on an exterior particle with mass m and position vector $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = -\operatorname{grad} U(\mathbf{r}) = -\frac{GmM}{r^2}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction,

$$U(\mathbf{r}) = -\frac{GmM}{r}$$

is the gravitational potential energy of the particle, and

$$M = \int_B \rho(\mathbf{r}')\, dV$$

is the total mass of the body.

Section 3: Orbits

1. A **central force F**, acting from the origin on a particle with position vector $\mathbf{r}$, has the form

$$\mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r},$$

where $g(\mathbf{r}) = g(x, y, z)$ is a scalar function of position. If the particle has mass m and is not acted upon by any other force, then it has constant angular momentum $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$. If the unit vector $\mathbf{k}$ is chosen in the direction of $\mathbf{L}$, then

(i) the motion of the particle is confined to the (x, y)-plane;

(ii) $\mathbf{L} = L_z\mathbf{k}$, where $L_z = mr^2\dot{\theta}$ is constant. The gravitational force due to a spherically symmetric body whose centre is fixed at the origin is a central force. In this case, the constancy of L_z is equivalent to Kepler's second law.

2. An **isotropic** central force has the form

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r,$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, and $\mathbf{e}_r = \mathbf{r}/r$. For a particle of mass m acted upon by such a force, the radial equation of motion is

$$m\ddot{r} = f(r) + \frac{L_z^2}{mr^3},$$

which may be integrated (analytically or numerically) to give $r(t)$. The gravitational force is isotropic, with $f(r) = -GmM/r^2$, where M is the central mass and G is the gravitational constant. (It is assumed that M is much larger than m, in order that the central mass should remain fixed at the origin.)

3. For the particle moving under gravity as described above, the total mechanical energy E is constant. It may be expressed as

(i) $$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r),$$

where $U(r) = -GmM/r$ is the (actual) potential energy of the particle;

(ii) $$E = \tfrac{1}{2}m\dot{r}^2 + U^{\text{eff}}(r),$$

where

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GmM}{r}$$

is the effective potential energy. Orbits are unbound if $E \geq 0$, and bound if $U^{\text{eff}}_{\min} \leq E < 0$, where $U^{\text{eff}}_{\min}$ is the minimum value of $U^{\text{eff}}(r)$.

4. The equation of gravitational orbit for the particle described above is

$$\frac{l}{r} = 1 + e\cos\theta,$$

where the semi-latus rectum l is given by

$$l = \frac{L_z^2}{m^2MG}.$$

If $0 \leq e < 1$ then the orbit is an ellipse with eccentricity e, showing that Kepler's first law is satisfied. If $e \geq 1$ then the motion is unbound.

In terms of the eccentricity and the semi-latus rectum, the total mechanical energy is

$$E = \frac{GmM}{2l}(e^2 - 1).$$

5. Kepler's third law is also a consequence of this Newtonian model, since the period T of an elliptical orbit and its semi-major axis a are related by the equation

$$T^2 = ka^3, \quad \text{where } k = \frac{4\pi^2}{MG}.$$

Section 4: Central forces and conservation laws

1. If a scalar function $\phi(\mathbf{r})$ depends upon position only through the radial distance $|\mathbf{r}| = r = (x^2 + y^2 + z^2)^{1/2}$, so that $\phi(\mathbf{r}) = \phi(r)$, then the gradient of ϕ can be written as

$$\operatorname{grad}\phi = \frac{d\phi}{dr}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction.

2. Hence if $\mathbf{F}$ is an isotropic central force, that is,

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r$$

for some function f, then $\mathbf{F}$ is conservative, with a potential energy function $U(r)$ such that

$$f(r) = -\frac{dU}{dr}.$$

The motion of a particle in this force field is planar, and has constant total mechanical energy

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r).$$

Unit 31 Fourier analysis

Section 1: What Fourier analysis is used for

1. A function $f(t)$ is said to be **periodic**, with **period** T, if

$$f(t) = f(t + nT) \quad \text{for all } t \quad (n = \pm 1, \pm 2, \pm 3, \ldots);$$

that is, the graph of $f(t)$ consists of repeating blocks of length T. We usually take T as the size of the smallest such block, but any multiple of this also satisfies the definition of a period.

2. If the response of a mechanical system to an input $F_0 \sin \omega t$ is $A \sin(\omega t + \phi)$, we call A/F_0 the **relative response** and ϕ the **phase shift**. The **frequency response** of the system is described by giving A and ϕ as functions of ω. For a **linear system** (i.e. one that can be modelled by a linear differential equation whose input is the right-hand side of the equation and whose output is the unknown function), the *principle of superposition* (see *Unit 6*) enables us to find the output produced by any input that is a linear combination of sinusoidal functions. That is, if $C_1, C_2, \ldots$ are constants and $F_1(t), F_2(t), \ldots$ are sinusoidal functions, and if the input $F_r(t)$ acting alone produces output $x_r(t)$, then the input $\sum_{r=1}^{n} C_r F_r(t)$ produces the output $\sum_{r=1}^{n} C_r x_r(t)$.

3. The sum of a number of sinusoidal terms of different frequencies is a non-sinusoidal function and in general is also non-periodic; but if the frequencies of the sinusoids are all multiples of ω, then their sum will be periodic and will have period $2\pi/\omega$. In this case we call ω the **fundamental angular frequency**.

Section 2: Fourier series for periodic functions

1. If $f(t)$ is a periodic function with period T, then the **Fourier series** for $f(t)$ has the form

$$F(t) = M + \sum_{n=1}^{\infty} (A_n \sin n\omega t + B_n \cos n\omega t),$$

where $\omega = 2\pi/T$ is the fundamental angular frequency. The constants in this series are given by

$$M = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt,$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin n\omega t\,dt \quad (n = 1, 2, 3, \ldots),$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos n\omega t\,dt \quad (n = 1, 2, 3, \ldots).$$

2. A function is said to be **continuous** if its graph is an unbroken curve. A function is **piecewise continuous** if its graph consists of continuous pieces, with only a finite number of such pieces on any finite part of the domain and each piece having a well-defined end-point.

3. *Theorem 2.3*: If a periodic function and its derivative are piecewise continuous, then the function is equal to its Fourier series for all values of t at which there is no gap in the graph.

Section 3: Some outstanding points

1. A function $f(t)$ is said to be **even** if

$$f(t) = f(-t) \quad (\text{for all } t).$$

The graph of an even function is symmetric under reflection in the vertical axis. The Fourier series for an even function contains no sine terms.

An **odd** function $f(t)$ is one satisfying

$$f(t) = -f(-t) \quad (\text{for all } t).$$

The graph of an odd function is symmetric under a rotation about the origin through 180°. The Fourier series for an odd function consists of sine terms only.

2. We can often obtain useful approximations to a given periodic function $f(t)$ by truncating its Fourier series, i.e. using a limited number of terms.

Section 4: Extending the scope

1. The theory described in this unit is not restricted to periodic functions of the variable t; any variable, for example x, may be used.

2. A non-periodic function whose domain is a finite interval of the t-axis can be represented by a Fourier series; we do this by extending the definition of the function to the entire t-axis in such a way that the extended function is periodic, and then computing the Fourier series for the extended function. This series then represents the original function on its own restricted domain.

If the Fourier series is required to have sine (or cosine) terms only, it is necessary to use a periodic extension which is odd (or even).

Unit 32 Partial differential equations

Section 1: Introduction and definitions

1. A **partial differential equation** is an equation involving one or more of the partial derivatives of some unknown function of two or more variables, and possibly also the value of the function itself and the values of the independent variables. (In contrast, an **ordinary differential equation** is a differential equation where the unknown is a function of just *one* variable, and which therefore does not involve partial derivatives.) Solutions of partial differential equations may be required to satisfy **boundary conditions** on the boundary of the domain of the unknown function. If one of the independent variables represents time, a condition imposed at the initial time is usually called an **initial condition** rather than a boundary condition.

2. An equation of the form

$$a(x,t)\frac{\partial^2 U}{\partial x^2} + b(x,t)\frac{\partial^2 U}{\partial x \partial t} + c(x,t)\frac{\partial^2 U}{\partial t^2} + d(x,t)\frac{\partial U}{\partial x} + e(x,t)\frac{\partial U}{\partial t} + f(x,t)U = g(x,t),$$

where $U(x,t)$ is the unknown function and at least one of the given functions a, b, c is not the zero function, is called a **linear second-order partial differential equation**. If in addition $g(x,t)$ is the zero function, we say the equation is **homogeneous**; if the functions $a, b, \ldots, f$ are all constants, we say the equation has **constant coefficients**.

3. *Theorem 1*: If u and v are two solutions of any homogeneous linear partial differential equation, and A and B are arbitrary constants, then $Au + Bv$ is also a solution.

Section 2: Properties of coiled springs and elastic strings

1. A spring is in a **state of tension** if it is extended, and in a **state of compression** if it is compressed. The **modulus of elasticity** k of a spring is defined by

$$k = \frac{K}{l_0},$$

where K is its stiffness (denoted by k in *Unit 7*) and l_0 is its natural length. If a spring, or string, is in a state of tension, then for every point P along its length we define the **tension** T at P to be the magnitude of the force exerted by the part of the spring to one side of P on the part to the other side of P. The value of T is always positive for a spring or string in a state of tension.

2. For a spring stretched between two fixed points A, B, we define the **reference state** as the configuration which makes the tension the same all along the spring. For a general state of the spring, still confined to the line AB, the tension T at a point P on the spring is given by

$$T = (T_0 + k)\frac{ds}{dx} - k,$$

where

(i) T_0 is the tension in the reference state,

(ii) k is the modulus of elasticity,

(iii) x is the position of P in the reference state, measured with respect to an x-axis lying along AB,

(iv) $s(x)$ is the position of P in the general state.

If the spring is moving, so that s depends on the time t as well as on x, then $\dfrac{ds}{dx}$ in the above formula is replaced by $\dfrac{\partial s}{\partial x}(x,t)$.

3. A spring whose mass is not negligible is called a **heavy spring**. Vibrations of a stretched heavy spring in which all motion is along the direction of the spring itself are called **longitudinal vibrations**; vibrations in which the motion is at right angles to this direction are called **transverse vibrations**. Longitudinal vibrations can be modelled by the equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}, \tag{1}$$

where

(i) $U = s - x$,

(ii) $c^2 = (T_0 + k)/m$,

(iii) m is the mass of the spring per unit length in the reference state.

If the ends of the spring are fixed at the points corresponding to $x = 0$ and $x = L$, where L is given, the **boundary conditions** are

$$\left.\begin{aligned} U(0,t) &= 0 \\ U(L,t) &= 0 \end{aligned}\right\} \text{ for all } t. \tag{2}$$

Equation (1) is called the **wave equation** and the constant c in it is called the **wave speed**.

Section 3: The method of separation of variables

1. The method of **separation of variables**, applied to the wave equation (1) with boundary conditions (2), is to look for solutions of the form

$$U(x,t) = X(x)T(t), \tag{3}$$

where X, T are functions (of one variable) to be found. Substituting into the wave equation (1) and dividing by $X(x)T(t)$ gives

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \mu, \tag{4}$$

where μ is a constant. So the function X satisfies the ordinary differential equation

$$X''(x) = \mu X(x). \tag{5}$$

The boundary conditions (2) imply

$$X(0) = X(L) = 0. \tag{6}$$

For $\mu \geq 0$ the only function X satisfying (5) and (6) is the zero function, leading to the trivial solution $U = 0$ of the original partial differential equation. For $\mu < 0$ a non-trivial solution is possible:

$$X(x) = F \sin hx,$$

where $\mu = -h^2$ $(h > 0)$ and μ is chosen so that $\sin hL = 0$, i.e.

$$h = \frac{n\pi}{L},$$

where n is a positive integer. With this choice of μ, when we have solved the equation

$$T'' = c^2 \mu T$$

implied by (4), the resulting solution $U(x,t) = X(x)T(t)$ takes the form

$$U_n(x,t) = \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right),$$

where $n = 1, 2, 3, \ldots$.

2. A more general solution of the partial differential equation is (by Theorem 1):

$$U(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right). \tag{7}$$

If the solution is required to satisfy **initial conditions** of the form

$$U(x,0) = \phi(x)$$
$$= \text{initial position of point } x \text{ on spring}, \tag{8a}$$

$$\frac{\partial U}{\partial t}(x,0) = \psi(x)$$
$$= \text{initial velocity of point } x \text{ on spring}, \tag{8b}$$

then the coefficients A_n and B_n can be found from the Fourier sine series expansions for the initial position and velocity functions $\phi(x)$ and $\psi(x)$ defined in Equations (8a) and (8b).

Section 4: Transverse vibrations of a stretched string

1. Transverse vibrations of a heavy spring or string can also be modelled by the wave equation, though the value of the wave speed c is not the same as for longitudinal vibrations.

2. A boundary condition with the property that if two functions U_1 and U_2 satisfy it then so do all linear combinations $c_1U_1 + c_2U_2$ is said to be **homogeneous**; otherwise it is **non-homogeneous**.

3. The wave equation (1) with the non-homogeneous set of boundary conditions

$$U(0,t) = 0, \tag{9a}$$

$$U(L,t) = R\sin\omega t, \tag{9b}$$

with ω a constant, can be used to model transverse vibrations of a stretched string, one end of which is fixed and the other constrained to move sinusoidally.

For this problem, separation of variables gives the particular solution

$$U_0(x,t) = \frac{R\sin(\omega x/c)}{\sin(\omega L/c)}\sin\omega t.$$

The general solution is

$$U(x,t) = U_0(x,t) + V(x,t),$$

where $V(x,t)$ stands for the right-hand side of Equation (7). If initial conditions are given as well, the constants A_n, B_n can be determined using Fourier series, as in Section 3.

9 Index of definitions

A reference in this index beginning with the letter 'p' is to a particular page of this Handbook.

A reference consisting of only numbers refers to the unit-by-unit summaries in Section 8 of this Handbook. Normally, the reference consists of two or three numbers, of which the first describes the unit, the second the section, and the third (where applicable) the numbered item, for those sections which are split into items. (For example, 29.3.4 means Item 4 of Section 3 of the entry for *Unit 29.*)

A number on its own refers to the introductory part of the entry. (For example, 6 means the Introduction of the entry for *Unit 6.*)

10 Notation

The following symbols have standard meanings throughout the course.

i	square root of -1 (see 5.1)
g	magnitude of acceleration due to gravity (approximately $9.81\,\mathrm{m\,s^{-2}}$) (see 4.3.2)
sin, cos, tan	p. 8
sec, cosec, cot	p. 8
arcsin, arccos, etc.	p. 9
grad, curl	see *Unit 26*
lim	p. 11
$\log_e x$	the logarithm of x to the base $e = 2.718\,28\ldots$
$\simeq$	approximately equals
!	factorial
a/bc	$a \div (b \times c)$
$\sqrt{x}$	the *positive* square root of x
$x^{1/n}$	nth root of number x
x^{-n}	$1/x^n$
$\lvert x\rvert$	modulus of real number x $(= \sqrt{x^2})$
$\lvert z\rvert$	modulus of complex number z
$\overline{z}$	complex conjugate of complex number z
$\lvert\mathbf{v}\rvert$ or v	modulus or length of vector $\mathbf{v}$
$\hat{\mathbf{v}}$	unit vector along $\mathbf{v}$
$y(x)$, $f(x)$	see p. 11
f'	derived function of function f (p. 13)
f''	derived function of f' (p. 13)
$f^{(n)}$	nth derivative of f (p. 13)
$\dfrac{dy}{dx}$	first derivative, i.e. $f'(x)$ where $y = f(x)$
$\dot{x}$	$\dfrac{dx}{dt}$, i.e. $f'(t)$ where $x = f(t)$
$\ddot{x}$	$\dfrac{d^2x}{dt^2}$, i.e. $f''(t)$ where $x = f(t)$
y'	$\dfrac{dy}{dx}$, i.e. $f'(x)$ where $y = f(x)$
$\dfrac{d^2y}{dx^2}$ or y''	second derivative, i.e. $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$
$\dfrac{d^ny}{dx^n}$ or $y^{(n)}$	nth derivative
$\displaystyle\int f(x)\,dx$	indefinite integral of function $f(x)$
$\displaystyle\int_a^b f(x)\,dx$	definite integral of function $f(x)$ from $x = a$ to $x = b$
$\in$	belongs to
$[a, b]$	the set of all x such that $a \leq x \leq b$
(a, b)	the set of all x such that $a < x < b$
$\sum$	summation (see p. 5)
Ox	x-axis of coordinate system
(x, y)-plane	plane containing x- and y-axes

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